

CPUC Quantum Game Theory: A Semialgebraic Geometric Framework for Quantum Strategic Behavior

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Abstract

We introduce CPUC quantum game theory, a semialgebraic geometric framework in which a structured family of completely positive and trace-preserving (CPTP) quantum channels provides the parameter space for quantum strategic interaction. The central object is the cosine-product unital channel (CPUC), whose Bloch representation is

$$T(X, Y, Z) = \text{diag}(X, Y, Z), \quad X = \cos \theta_x \cos \theta_z, \quad Y = \cos \theta_y \cos \theta_z, \quad Z = \cos \theta_x \cos \theta_y.$$

Its image is a compact semialgebraic subset of the diagonal unital qubit-channel space. We regard this image as a geometric parameter space for a family of quantum games

$$p = (X, Y, Z) \mapsto G_p.$$

We prove that, after algebraization, the equilibrium correspondence of finite-player quantum games with compact semialgebraic strategy spaces and semialgebraic payoff functions is semialgebraic and admits finite semialgebraic stratification. We then distinguish the geometric four-lobe CPUC decomposition from the genuine strategic discriminant.

For the standard maximally entangled Eisert–Wilkins–Lewenstein (EWL) Prisoner’s Dilemma with canonical payoffs

$$(T, R, P, S) = (5, 3, 1, 0),$$

we derive an explicit principal best-response discriminant for the maximally quantum strategy

$$Q = U(0, \pi/2).$$

For the symmetric CPUC channel, the relevant channel dependence reduces to the invariant coordinates

$$A = XY, \quad B = Z^2,$$

and the principal Q -branch transition surface is

$$\boxed{P_{\Delta, Q}(X, Y, Z) = Z^2 - 7XY = 0.}$$

Thus a three-dimensional physical channel space carries an algebraic strategic phase boundary. We emphasize that this surface is a principal best-response component and is not, without further elimination, the complete Nash discriminant. The framework therefore provides a bridge between quantum-channel geometry, Nash equilibrium theory, real algebraic geometry, and channel-adaptive quantum control.

1 Introduction

Quantum game theory studies strategic interaction in which players may use quantum states, quantum operations, entanglement, and measurements as strategic resources. The Eisert–Wilkens–Lewenstein (EWL) construction demonstrated that replacing classical strategies by quantum strategies can qualitatively modify the equilibrium structure of classical games [2, 3, 4].

A natural geometric question is whether the physical quantum channel can itself be incorporated into the mathematical structure of a game. In ordinary game theory, environmental parameters are often treated as external data. In a noisy quantum game, however, the channel directly changes the payoff map and therefore changes the strategic landscape. We propose to treat a structured family of quantum channels as a parameter space for the game itself.

The central object is the Cosine-Product Unital Channel (CPUC) [1]. Its three Bloch contraction parameters are generated from three angles:

$$X = \cos \theta_x \cos \theta_z, \quad Y = \cos \theta_y \cos \theta_z, \quad Z = \cos \theta_x \cos \theta_y. \quad (1)$$

The resulting image is a compact semialgebraic subset $\mathcal{M}_{\text{CPUC}} \subset \mathbb{R}^3$. Its nonzero points occupy four sign-compatible sectors, which provide a useful coarse geometric classification.

The purpose of CPUC quantum game theory is not merely to introduce another coordinate system for channels. Rather, we regard

$$p = (X, Y, Z) \in \mathcal{M}_{\text{CPUC}}$$

as a physical environment parameter and a quantum game as a family

$$\mathcal{G}_{\text{CPUC}} = \{G_p : p \in \mathcal{M}_{\text{CPUC}}\}. \quad (2)$$

This produces the conceptual chain

quantum channel $\rightarrow$ CPUC geometry $\rightarrow$ quantum game $\rightarrow$ equilibrium correspondence $\rightarrow$ strategic stratification.
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The principal mathematical observation is that algebraization converts the relevant equilibrium conditions into semialgebraic conditions. Hence Tarski–Seidenberg projection and semialgebraic stratification provide a natural language for identifying regions in channel space on which the qualitative equilibrium structure is constant. The four CPUC lobes are only a coarse physical decomposition; the equilibrium discriminant refines that decomposition.

The paper has four main contributions.

- (i) We define the CPUC-parametrized quantum-game family and characterize the CPUC channel space as a compact semialgebraic subset of the diagonal unital qubit-channel space.
- (ii) We formulate the equilibrium correspondence as a semialgebraic object and establish a general finite-stratification result for semialgebraic quantum games.
- (iii) We distinguish physical channel boundaries from strategic discriminants and formulate the complete discriminant as the union of interior degeneracy, boundary-collision, and global best-response components.
- (iv) For the maximally entangled EWL Prisoner’s Dilemma, we explicitly derive the principal maximally quantum best-response discriminant

$$P_{\Delta, Q}(X, Y, Z) = Z^2 - 7XY,$$

thereby exhibiting an explicit algebraic strategic phase boundary inside a three-dimensional quantum-channel parameter space.

The result should be interpreted carefully. The four CPUC lobes are not asserted to be equilibrium-invariant for arbitrary games. Likewise, $Z^2 - 7XY = 0$ is a principal discriminant component for the Q branch; the complete Nash discriminant may contain additional components. These distinctions are essential to the mathematical scope of the framework.

2 Quantum Games and CPTP Channels

A quantum state is a density operator

$$\rho \geq 0, \quad \text{Tr } \rho = 1.$$

A physical quantum operation is represented by a completely positive and trace-preserving (CPTP) map

$$\Phi : \rho \mapsto \Phi(\rho).$$

Complete positivity requires

$$(\Phi \otimes I_k)(A) \geq 0$$

for every positive operator A and every ancillary dimension k , while trace preservation requires

$$\text{Tr } \Phi(\rho) = \text{Tr } \rho.$$

For a two-player quantum game with initial state ρ_0 , local strategic operations $U_1(s_1)$ and $U_2(s_2)$, and physical channels $\Phi_{p,1}$ and $\Phi_{p,2}$, the final state can be written as

$$\rho_f(s_1, s_2; p) = (\Phi_{p,1} \otimes \Phi_{p,2}) \left[(U_1(s_1) \otimes U_2(s_2)) \rho_0 (U_1(s_1) \otimes U_2(s_2))^\dagger \right]. \quad (3)$$

If R_i is the payoff observable of player i , then

$$u_i(s_1, s_2; p) = \text{Tr}[R_i \rho_f(s_1, s_2; p)]. \quad (4)$$

The essential separation is

$p \in \mathcal{M}_{\text{CPUC}}$ is an environmental/channel parameter, $s_i \in S_i$ is a strategic variable.

The channel coordinates need not themselves be players' actions. They parameterize the physical environment in which the strategic interaction takes place.

3 Cosine-Product Unital Channels [1]

Consider a unital qubit channel acting on the Bloch vector

$$\rho = \frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma}), \quad \mathbf{r} \mapsto T\mathbf{r},$$

with

$$T(X, Y, Z) = \text{diag}(X, Y, Z).$$

The CPUC parameterization is

$$X = \cos \theta_x \cos \theta_z, \quad Y = \cos \theta_y \cos \theta_z, \quad Z = \cos \theta_x \cos \theta_y. \quad (5)$$

Set

$$a = \cos \theta_x, \quad b = \cos \theta_y, \quad c = \cos \theta_z.$$

Then

$$(X, Y, Z) = (ac, bc, ab). \quad (6)$$

It follows immediately that

$$XYZ = a^2 b^2 c^2 \geq 0. \quad (7)$$

Furthermore,

$$\frac{|XY|}{|Z|} = c^2 \leq 1,$$

whenever $Z \neq 0$, and cyclically. Hence

$$|XY| \leq |Z|, \quad |XZ| \leq |Y|, \quad |YZ| \leq |X|. \quad (8)$$

Define

$$\mathcal{M}_{\text{CPUC}} = \{(X, Y, Z) \in \mathbb{R}^3 : XYZ \geq 0, |XY| \leq |Z|, |XZ| \leq |Y|, |YZ| \leq |X|\}. \quad (9)$$

Proposition 3.1. *The image of the cosine-product map*

$$(a, b, c) \in [-1, 1]^3 \mapsto (ac, bc, ab)$$

is exactly $\mathcal{M}_{\text{CPUC}}$. In particular, $\mathcal{M}_{\text{CPUC}}$ is compact and semialgebraic.

Proof. The inclusion of the image in $\mathcal{M}_{\text{CPUC}}$ follows from (7)–(8). Conversely, suppose $(X, Y, Z) \in \mathcal{M}_{\text{CPUC}}$ with $XYZ \neq 0$. Define

$$a^2 = \frac{XZ}{Y}, \quad b^2 = \frac{YZ}{X}, \quad c^2 = \frac{XY}{Z}.$$

The sign condition $XYZ \geq 0$ makes the three ratios nonnegative, and the inequalities in (8) imply

$$0 \leq a^2, b^2, c^2 \leq 1.$$

Choose signs of a, b, c so that

$$ac = X, \quad bc = Y, \quad ab = Z.$$

The cases in which one or more coordinates vanish follow similarly: if, say, $Z = 0$ then the defining inequalities force $XY = 0$ as well, so at least one of X, Y also vanishes; taking the corresponding factor among a, b, c to be 0 (e.g. $c = 0$ if $X = Z = 0$) forces the two coordinates built from it to vanish automatically, while the remaining coordinate is attained by choosing the two remaining factors freely in $[-1, 1]$ with the required product. Iterating this argument over the finitely many sign patterns of vanishing coordinates shows every boundary point of $\mathcal{M}_{\text{CPUC}}$ is attained. Hence every point of $\mathcal{M}_{\text{CPUC}}$ is attained. $\square$

The CPUC construction is therefore a structured subfamily of the diagonal unital qubit-channel space rather than an arbitrary subset of $\mathbb{R}^3$.

4 Pauli-Channel Representation

Every diagonal unital qubit channel can be represented as a Pauli channel

$$\Phi_{X,Y,Z}(\rho) = p_I \rho + p_X \sigma_x \rho \sigma_x + p_Y \sigma_y \rho \sigma_y + p_Z \sigma_z \rho \sigma_z, \quad (10)$$

where

$$p_I = \frac{1 + X + Y + Z}{4}, \quad (11)$$

$$p_X = \frac{1 + X - Y - Z}{4}, \quad (12)$$

$$p_Y = \frac{1 - X + Y - Z}{4}, \quad (13)$$

$$p_Z = \frac{1 - X - Y + Z}{4}. \quad (14)$$

These satisfy

$$p_I + p_X + p_Y + p_Z = 1.$$

Complete positivity is equivalent to

$$p_I, p_X, p_Y, p_Z \geq 0.$$

Thus the complete-positivity region of diagonal unital qubit channels is the Pauli tetrahedron.

The CPUC set is a structured semialgebraic subregion of this physical parameter space [1]:

$$\mathcal{M}_{\text{CPUC}} \subseteq \mathcal{T}_{\text{Pauli}}.$$

The distinction is important. The Pauli tetrahedron describes physical admissibility, whereas the CPUC construction selects a particular three-dimensional family generated by the cosine-product map [1].

The Pauli probabilities can also be used to interpret the CPUC coordinates operationally. For example, the total computational-basis bit-flip probability is

$$q = p_X + p_Y = \frac{1 - Z}{2}. \quad (15)$$

This identity will be central for the restricted classical-action Prisoner's Dilemma and illustrates that a particular game can probe only a lower-dimensional projection of the full channel geometry.

5 CPUC Geometry and Four-Lobe Structure [1]

Because $XYZ \geq 0$, the nonzero CPUC points occupy four sign-compatible sectors:

$$L_I = \{X > 0, Y > 0, Z > 0\}, \quad (16)$$

$$L_X = \{X > 0, Y < 0, Z < 0\}, \quad (17)$$

$$L_Y = \{X < 0, Y > 0, Z < 0\}, \quad (18)$$

$$L_Z = \{X < 0, Y < 0, Z > 0\}. \quad (19)$$

Let Σ denote the lower-dimensional set containing coordinate-zero and boundary strata. Then

$$\mathcal{M}_{\text{CPUC}} = L_I \cup L_X \cup L_Y \cup L_Z \cup \Sigma. \quad (20)$$

These four sectors are naturally associated with the four Pauli directions. This association is geometric and does not imply that every point in a sector represents a pure Pauli operation.

The four sectors admit a useful discrete coarse-graining by the complete graph

$$V(K_4) = \{L_I, L_X, L_Y, L_Z\}.$$

The K_4 representation is a combinatorial summary of the four-sector geometry; it is not, by itself, a statement about equilibrium topology.

The distinction between physical and strategic geometry is fundamental:

$\partial\mathcal{M}_{\text{CPUC}}$ is a physical channel boundary,

whereas

Δ_{game} is a strategic equilibrium boundary.

There is no general reason for these two sets to coincide.

6 CPUC-Parametrized Quantum Games

Definition 6.1 (CPUC quantum game). *A CPUC quantum game is a parametrized family*

$$\mathcal{G}_{\text{CPUC}} = \{G_p : p \in \mathcal{M}_{\text{CPUC}}\},$$

where the physical quantum channel(s) entering the game depend on $p = (X, Y, Z)$.

For a two-player game with strategy spaces S_1, S_2 , write

$$s = (s_1, s_2) \in S = S_1 \times S_2.$$

A general payoff has the form

$$u_i(s_1, s_2; p) = \text{Tr} \left[R_i(\Phi_{p,1} \otimes \Phi_{p,2}) \left(U(s) \rho_0 U(s)^\dagger \right) \right]. \quad (21)$$

The channel parameter and strategic variable play different roles:

$$p \in \mathcal{M}_{\text{CPUC}} \quad \text{versus} \quad s_i \in S_i.$$

A channel-adaptive policy is a map

$$\pi_i : \mathcal{M}_{\text{CPUC}} \rightarrow S_i, \quad s_i = \pi_i(p).$$

This provides the abstract feedback architecture

$$\boxed{\text{channel estimation} \longrightarrow \text{CPUC coordinate} \longrightarrow \text{strategic regime} \longrightarrow \text{adaptive action.}} \quad (22)$$

7 Nash Equilibrium Correspondence

For each $p \in \mathcal{M}_{\text{CPUC}}$, define the Nash equilibrium set

$$\text{NE}(G_p) = \left\{ s \in S : s_i \in \arg \max_{t_i \in S_i} u_i(t_i, s_{-i}; p) \text{ for all } i \right\}.$$

The equilibrium graph is

$$\mathcal{E} = \{(p, s) \in \mathcal{M}_{\text{CPUC}} \times S : s \in \text{NE}(G_p)\}. \quad (23)$$

The central question is how $\text{NE}(G_p)$ changes as p moves through $\mathcal{M}_{\text{CPUC}}$. A geometric channel decomposition is useful only if it can be refined into regions on which the qualitative equilibrium structure is constant.

For isolated interior equilibria described locally by

$$F(s, p) = 0,$$

the implicit-function theorem gives local continuation whenever

$$\det D_s F(s, p) \neq 0.$$

A principal stationary discriminant is therefore

$$\Delta_{\text{sing}} = \{p \in \mathcal{M}_{\text{CPUC}} : \exists s, F(s, p) = 0, \det D_s F(s, p) = 0\}. \quad (24)$$

However, the complete Nash discriminant must also account for boundary collisions and global best-response transitions. We therefore define

$$\boxed{\Delta_{\text{Nash}} = \Delta_{\text{int}} \cup \Delta_{\partial} \cup \Delta_{\text{BR}}.} \quad (25)$$

Here Δ_{int} denotes interior equilibrium degeneracy, Δ_{∂} denotes collisions with active strategy-space boundaries, and Δ_{BR} denotes equality of competing global best-response values.

The connected components of

$$\mathcal{M}_{\text{CPUC}} \setminus \Delta_{\text{Nash}}$$

are the natural candidates for invariant qualitative strategic regimes.

8 Algebraization of the EWL Game

The EWL strategy is

$$U(\theta, \phi) = \begin{pmatrix} e^{i\phi} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & e^{-i\phi} \cos(\theta/2) \end{pmatrix}, \quad (26)$$

with

$$0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq \frac{\pi}{2}.$$

Introduce algebraic variables

$$x = \cos(\theta/2), \quad y = \sin(\theta/2), \quad c = \cos \phi, \quad s = \sin \phi.$$

They satisfy

$$x^2 + y^2 = 1, \quad c^2 + s^2 = 1, \quad x, y, c, s \geq 0. \quad (27)$$

The EWL unitary entries become polynomial expressions in these variables and the phase factors can be represented through real and imaginary components. The resulting state and payoff functions are polynomial (or, equivalently, semialgebraic) functions on a compact semialgebraic domain.

For two players, write

$$s = (x_1, y_1, c_1, s_1, x_2, y_2, c_2, s_2).$$

Then the Nash conditions can be expressed by finitely many polynomial equalities and inequalities, together with the circle constraints (27).

This algebraization is the mechanism that connects the EWL game to real algebraic geometry. In particular, the projection of an algebraic or semialgebraic equilibrium set onto the channel variables remains semialgebraic by the Tarski–Seidenberg theorem.

9 Discriminants and Equilibrium Stratification

The strategic discriminant is the subset of channel space at which the qualitative equilibrium correspondence may change. For the EWL game, three mechanisms must be included:

- (1) an interior stationary equilibrium becomes degenerate;
- (2) an equilibrium reaches an active boundary of the strategy space;
- (3) two or more competing global best-response values become equal.

After algebraization, each mechanism is described by polynomial equalities and inequalities. Projection onto (X, Y, Z) therefore gives semialgebraic subsets.

Proposition 9.1 (Semialgebraic discriminant). *For a fixed finite-player quantum game whose strategy spaces are compact semialgebraic and whose payoff functions are semialgebraic in both strategic and channel variables, the equilibrium graph is semialgebraic. Consequently, the corresponding strategic discriminant is semialgebraic.*

Proof. Fix the algebraization of Section 8: each strategy space S_i is realized as a compact real-algebraic variety $\{(x_i, y_i, c_i, s_i) \in [-1, 1]^4 : x_i^2 + y_i^2 = 1, c_i^2 + s_i^2 = 1\}$, and each payoff $u_i(s; p)$ is a polynomial function of (s, p) , being a trace of a finite product of matrices whose entries are polynomial in the algebraized strategy variables and in (X, Y, Z) . In particular each S_i and each u_i is semialgebraic.

The Nash condition defines

$$\varphi(s, p) \equiv s \in S \wedge \bigwedge_{i=1}^n \forall t_i (t_i \in S_i \rightarrow u_i(t_i, s_{-i}; p) \leq u_i(s; p)),$$

a first-order formula in the language of ordered fields, with (s, p) free and t_i bound, whose atomic predicates ($t_i \in S_i$, the two polynomial inequalities) are all semialgebraic. By the Tarski–Seidenberg theorem [8, 15] — the projection theorem for semialgebraic sets, equivalently quantifier elimination for real closed fields — every set defined by such a formula is semialgebraic, and is in fact equivalent over $\mathbb{R}$ to a quantifier-free Boolean combination of finitely many polynomial equalities and inequalities in the free variables alone. We do not exhibit this quantifier-free form; its existence is all Proposition 9.1 requires. Hence

$$E = \{(p, s) \in M_{CPUC} \times S : \varphi(s, p)\}$$

is semialgebraic, which is the content of Proposition 11.1 below and is used here as an intermediate step.

The three discriminant mechanisms are each obtained from E (or from auxiliary semialgebraic data) by the same projection principle:

- Δ_{int} : the stationarity system $F(s, p) = 0$ for interior critical points is polynomial after introducing Lagrange multipliers for the algebraic constraints $x_i^2 + y_i^2 - 1 = 0$, $c_i^2 + s_i^2 - 1 = 0$; the degeneracy condition $\det D_s F(s, p) = 0$ is polynomial in the same variables. Thus Δ_{int} is the image under the coordinate projection π_p of the real algebraic set $\{(s, p, \lambda) : F = 0, \det D_s F = 0\}$, hence semialgebraic by Tarski–Seidenberg.
- Δ_{∂} : boundary equilibria are characterized by the same stationarity system restricted to the algebraic boundary strata of S (e.g. $y_i = 0$), together with a one-sided (KKT-type) inequality replacing the interior first-order condition; again a semialgebraic system, and $\Delta_{\partial} = \pi_p$ of its solution set.
- Δ_{BR} : equality of two competing best-response branches is the condition $u_i(\text{branch}_a; p) = u_i(\text{branch}_b; p)$, where each branch is itself cut out by a semialgebraic (possibly existentially quantified, if not given in closed form) condition on s ; the equality locus is again semialgebraic by Tarski–Seidenberg applied to the corresponding formula.

Since semialgebraic sets form a Boolean algebra (closed under finite union, intersection, and complement — this is part of the content of the Tarski–Seidenberg theorem, not a separate assumption), $\Delta_{\text{Nash}} = \Delta_{\text{int}} \cup \Delta_{\partial} \cup \Delta_{BR}$ is semialgebraic. $\square$

The discriminant decomposes channel space into

$$\mathcal{M}_{CPUC} = (\mathcal{M}_{CPUC} \setminus \Delta_{\text{Nash}}) \cup \Delta_{\text{Nash}}.$$

A further finite semialgebraic stratification can be chosen compatible with the equilibrium graph and the discriminant. On regular strata, Hardt-type semialgebraic triviality provides a precise formulation of qualitative constancy after the necessary refinement.

10 The Three-Dimensional EWL Discriminant

10.1 Standard EWL Prisoner’s Dilemma

We now compute an explicit strategic discriminant for the standard maximally entangled EWL Prisoner’s Dilemma.

Let

$$D = i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and define the maximally entangling gate

$$J = \frac{1}{\sqrt{2}} (I \otimes I + iD \otimes D). \quad (28)$$

The initial state is

$$|\psi_0\rangle = J|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle). \quad (29)$$

For player i ,

$$U_i = U(\theta_i, \phi_i)$$

with U given by (26). The strategic state before the channel is

$$\rho_{\text{str}} = (U_1 \otimes U_2) J |00\rangle\langle 00| J^\dagger (U_1 \otimes U_2)^\dagger. \quad (30)$$

The symmetric CPUC channel is

$$\Phi_{X,Y,Z}(\rho) = p_I \rho + p_X \sigma_x \rho \sigma_x + p_Y \sigma_y \rho \sigma_y + p_Z \sigma_z \rho \sigma_z,$$

with the probabilities of Section 4. The final EWL state is

$$\rho_f = J^\dagger (\Phi_{X,Y,Z} \otimes \Phi_{X,Y,Z})(\rho_{\text{str}}) J. \quad (31)$$

For the canonical Prisoner's Dilemma,

$$(T, R, P, S) = (5, 3, 1, 0),$$

the payoff observables are

$$R_A = 3 |00\rangle\langle 00| + 0 |01\rangle\langle 01| + 5 |10\rangle\langle 10| + 1 |11\rangle\langle 11|, \quad (32)$$

$$R_B = 3 |00\rangle\langle 00| + 5 |01\rangle\langle 01| + 0 |10\rangle\langle 10| + 1 |11\rangle\langle 11|. \quad (33)$$

Thus

$$u_A = \text{Tr}(R_A \rho_f), \quad u_B = \text{Tr}(R_B \rho_f).$$

10.2 Channel invariants

For the symmetric CPUC–EWL construction, direct Pauli-basis reduction shows that the channel dependence relevant to the principal quantum branch factors through

$$A = XY, \quad B = Z^2. \quad (34)$$

Thus the relevant payoff restriction can be written

$$u_i = \tilde{u}_i(\theta_1, \phi_1, \theta_2, \phi_2; A, B).$$

The corresponding quotient map is

$$\pi : \mathcal{M}_{\text{CPUC}} \rightarrow \mathbb{R}^2, \quad \pi(X, Y, Z) = (XY, Z^2). \quad (35)$$

Consequently, the associated discriminant is the pullback of a discriminant in the invariant (A, B) -plane:

$$\Delta_{\text{EWL}} = \pi^{-1}(\Delta_{A,B})$$

for the invariant sector under consideration.

The invariance gives the differential identity

$$\boxed{(X\partial_X - Y\partial_Y) P_\Delta(X, Y, Z) = 0.} \quad (36)$$

The principal polynomial is also even in Z :

$$P_\Delta(X, Y, Z) = P_\Delta(X, Y, -Z).$$

Remark 10.1. *The reduction to (XY, Z^2) is a statement about the symmetric EWL–CPUC payoff structure being considered here. It should not be assumed for arbitrary quantum games, asymmetric channels, or different entangling operators.*

10.3 Q -branch

The maximally quantum EWL strategy is

$$Q = U(0, \pi/2). \quad (37)$$

Fix player 2 at Q . For the canonical payoff tuple, the player-1 payoff on this unilateral slice reduces to

$$u_A(\theta_1, \phi_1; Q) = C(A, B) + \cos^2\left(\frac{\theta_1}{2}\right) \left[-2A \cos^2 \phi_1 + \frac{7A - B}{2} \right], \quad (38)$$

where $C(A, B)$ is independent of player 1's strategic variables.

For

$$A = XY > 0,$$

the phase-dependent term is maximized at

$$\phi_1 = \frac{\pi}{2}.$$

Consequently,

$$u_A(\theta_1, \pi/2; Q) = C(A, B) + \frac{7A - B}{2} \cos^2\left(\frac{\theta_1}{2}\right). \quad (39)$$

The sign of $7A - B$ therefore determines the best response:

$$7A - B > 0 \implies \theta_1 = 0, \quad (40)$$

$$7A - B < 0 \implies \theta_1 = \pi. \quad (41)$$

At

$$7A - B = 0$$

the competing branches have equal payoff.

10.4 Explicit discriminant

The transition condition is

$$B = 7A.$$

Using

$$A = XY, \quad B = Z^2,$$

we obtain

$$\boxed{P_{\Delta, Q}(X, Y, Z) = Z^2 - 7XY.} \quad (42)$$

The principal Q -branch discriminant surface is therefore

$$\boxed{\Delta_Q = \{(X, Y, Z) \in \mathcal{M}_{\text{CPUC}} : Z^2 = 7XY, XY > 0\}.} \quad (43)$$

The two sides have distinct best-response behavior:

$$Z^2 < 7XY, \quad XY > 0 \implies (Q, Q) \text{ is a strict best-response equilibrium,} \quad (44)$$

$$Z^2 > 7XY, \quad XY > 0 \implies (Q, Q) \text{ ceases to be a best-response equilibrium.} \quad (45)$$

At

$$Z^2 = 7XY$$

the Q branch is strategically degenerate.

Theorem 10.2 (Principal three-dimensional EWL discriminant). *For the standard maximally entangled EWL Prisoner's Dilemma with canonical payoffs*

$$(T, R, P, S) = (5, 3, 1, 0),$$

and identical symmetric CPUC channels on the two qubits, the principal maximally quantum best-response branch

$$Q = U(0, \pi/2)$$

has discriminant

$$P_{\Delta, Q}(X, Y, Z) = Z^2 - 7XY.$$

Equivalently, in the invariant coordinates

$$A = XY, \quad B = Z^2,$$

the transition is the line

$$B = 7A.$$

Proof. Equation (39) shows that, for $A > 0$, the optimal phase is $\phi_1 = \pi/2$ and the remaining strategic dependence is proportional to

$$(7A - B) \cos^2(\theta_1/2).$$

Thus the quantum branch is optimal for $7A - B > 0$, the competing boundary branch is optimal for $7A - B < 0$, and the transition occurs exactly at $7A - B = 0$. Substituting $A = XY$ and $B = Z^2$ gives

$$Z^2 - 7XY = 0.$$

□

10.5 PDE and geometric interpretation

Differentiating

$$P_{\Delta, Q} = Z^2 - 7XY$$

gives

$$(P_{\Delta, Q})_X = -7Y, \quad (P_{\Delta, Q})_Y = -7X, \quad (P_{\Delta, Q})_Z = 2Z.$$

Therefore

$$X(P_{\Delta, Q})_X - Y(P_{\Delta, Q})_Y = -7XY + 7XY = 0,$$

so

$$(X\partial_X - Y\partial_Y)P_{\Delta, Q} = 0. \tag{46}$$

The discriminant is invariant under

$$(X, Y, Z) \mapsto (e^t X, e^{-t} Y, Z)$$

and under

$$Z \mapsto -Z.$$

Its normal vector is

$$\nabla P_{\Delta, Q} = (-7Y, -7X, 2Z).$$

Let

$$p(t) = (X(t), Y(t), Z(t))$$

be a differentiable physical channel trajectory. A transverse crossing of the discriminant at $t = t_*$ satisfies

$$P_{\Delta,Q}(p(t_*)) = 0$$

and

$$\left. \frac{d}{dt} P_{\Delta,Q}(p(t)) \right|_{t=t_*} \neq 0.$$

Explicitly,

$$\boxed{-7Y\dot{X} - 7X\dot{Y} + 2Z\dot{Z} \neq 0.} \quad (47)$$

Thus $P_{\Delta,Q}$ gives a direct scalar criterion for crossing the principal strategic phase boundary.

The discriminant should not be confused with the physical boundary:

$$\Delta_Q \neq \partial \mathcal{M}_{\text{CPUC}}$$

in general. A strategic transition can occur at a physically admissible interior point of the channel space.

10.6 Scope and complete discriminant

The polynomial

$$P_{\Delta,Q} = Z^2 - 7XY$$

is the explicit principal best-response discriminant of the Q branch. It is not, without further elimination, the complete Nash discriminant.

The complete discriminant has the decomposition

$$\Delta_{\text{Nash}} = \Delta_{\text{int}} \cup \Delta_{\partial} \cup \Delta_{\text{BR}}.$$

Additional components may arise from other interior stationary branches, strategy-boundary equilibria, equilibrium collisions, or equality of competing global best-response values.

A complete algebraic computation is obtained by introducing

$$x_i = \cos(\theta_i/2), \quad y_i = \sin(\theta_i/2),$$

and

$$c_i = \cos \phi_i, \quad s_i = \sin \phi_i,$$

with

$$x_i^2 + y_i^2 = 1, \quad c_i^2 + s_i^2 = 1.$$

The stationary equations together with

$$\det D_s F = 0$$

define an algebraic ideal in the joint strategy-channel variables. Eliminating the strategy variables produces the stationary discriminant ideal in

$$\mathbb{R}[X, Y, Z].$$

The invariant reduction implies that the relevant symmetric factors may be represented as

$$\tilde{P}_{\Delta}(XY, Z^2).$$

The explicit Q -branch factor is

$$\boxed{Z^2 - 7XY}.$$

Thus the hierarchy

$$\boxed{\mathcal{M}_{\text{CPUC}} \longrightarrow (XY, Z^2) \longrightarrow P_{\Delta,Q} = Z^2 - 7XY \longrightarrow \text{strategic phase diagram}}$$

provides a concrete realization of channel-induced strategic stratification.

11 Semialgebraic Equilibrium Stratification

We now state the general structural result underlying the framework.

Proposition 11.1 (Semialgebraic equilibrium graph). *Let*

$$G_p, \quad p \in \mathcal{M}_{\text{CPUC}},$$

be a finite-player quantum game with compact semialgebraic strategy spaces and semialgebraic payoff functions. Then the equilibrium graph

$$\mathcal{E} = \{(p, s) : s \in \text{NE}(G_p)\}$$

is semialgebraic.

Proof. This restates the first step of Proposition 9.1’s proof in the general n -player setting; we give it independently for completeness. As above, semialgebraicity of S_i and of $u_i(\cdot; p)$ is inherited from the algebraization of the strategy spaces (Section 8) and from u_i being a polynomial trace functional of (s, p) .

The formula $\varphi(s, p)$ defining the Nash condition is a first-order sentence over $(\mathbb{R}, +, \times, <)$ with parameters, built from finitely many polynomial atomic predicates, one universal quantifier block per player ($\forall t_i \in S_i$), and finite Boolean combinations. The Tarski–Seidenberg theorem [8, 15] guarantees that any subset of $\mathbb{R}^N$ defined by such a formula — regardless of quantifier depth or alternation — is semialgebraic; the theorem’s content is precisely that quantifiers over the reals can always be eliminated in principle, yielding a logically equivalent quantifier-free (hence manifestly semialgebraic) description, though we make no use of the specific eliminated formula here. Applying this to φ with (p, s) as free variables gives that

$$E = \{(p, s) : \varphi(s, p)\}$$

is semialgebraic. □

Proposition 11.2 (Finite strategic stratification). *Under the hypotheses above, there exists a finite semialgebraic stratification*

$$\mathcal{M}_{\text{CPUC}} = \bigcup_{\alpha} S_{\alpha}$$

compatible with the projection of the equilibrium graph and with the relevant discriminant loci. After refining by the discriminant, the equilibrium correspondence is semialgebraically trivial over regular strata.

Proof. By Proposition 11.1, $E \subset \mathcal{M}_{\text{CPUC}} \times S$ is semialgebraic, and the coordinate projection $\pi : E \rightarrow \mathcal{M}_{\text{CPUC}}$, $\pi(p, s) = p$, is a continuous semialgebraic map (a linear projection restricted to a semialgebraic set is semialgebraic).

We invoke **Hardt’s semialgebraic triviality theorem** [16] — see also its treatment in Bochnak–Coste–Roy [17, Thm. 9.3.2] and, in the more general o-minimal setting that contains the semialgebraic case, van den Dries [18, Ch. 9]: for a continuous semialgebraic map $f : A \rightarrow \mathbb{R}^k$ between semialgebraic sets, there is a finite semialgebraic partition $\mathbb{R}^k = \bigsqcup_{\alpha} S_{\alpha}$ such that f is **semialgebraically trivial** over each S_{α} — that is, $f^{-1}(S_{\alpha})$ is semialgebraically homeomorphic to $S_{\alpha} \times f^{-1}(q_{\alpha})$ (for any fixed $q_{\alpha} \in S_{\alpha}$) via a homeomorphism commuting with f and the projection to S_{α} . Crucially, the theorem also states that this trivialization can be chosen **compatible** with any finite prescribed family of semialgebraic subsets of A and of $\mathbb{R}^k$: each such subset is a union of pieces $S_{\alpha} \times (\text{fixed subset of the fiber})$, respectively a union of strata S_{α} .

Apply this with $f = \pi$, $A = E$, target $\mathbb{R}^k \supset \mathcal{M}_{\text{CPUC}}$, and the prescribed family $\{\Delta_{\text{int}}, \Delta_{\partial}, \Delta_{BR}, \Delta_{\text{Nash}}\}$ from Proposition 9.1. This yields a finite semialgebraic stratification $\mathcal{M}_{\text{CPUC}} = \bigcup_{\alpha} S_{\alpha}$ such that:

1. π is semialgebraically trivial over each S_α , so the fiber $\pi^{-1}(p) = NE(G_p)$ has constant semialgebraic homeomorphism type — in particular constant dimension, number of connected components, and local structure — as p ranges over S_α ; 2. each S_α lies entirely inside Δ_{Nash} or entirely inside $M_{\text{CPUC}} \setminus \Delta_{\text{Nash}}$, by the compatibility clause of Hardt's theorem applied to Δ_{Nash} .

This is exactly the stratification asserted in the Proposition, and (2) is what licenses Definition 11.3: on any regular stratum $S_\alpha \subset M_{\text{CPUC}} \setminus \Delta_{\text{Nash}}$, the equilibrium correspondence is semialgebraically trivial by (1), hence in particular has constant qualitative type throughout S_α . $\square$

Definition 11.3 (Equilibrium-invariant stratum). *A connected stratum*

$$S_\alpha \subset \mathcal{M}_{\text{CPUC}} \setminus \Delta_{\text{Nash}}$$

is called equilibrium-invariant if the equilibrium correspondence has constant qualitative type throughout S_α .

For isolated nondegenerate equilibria, the implicit-function theorem gives local continuation. The global semialgebraic decomposition refines this local result into finitely many regular regimes.

The key conclusion is therefore

$\text{CPUC geometric sectors become strategic regimes only after refinement by the equilibrium discriminant.}$

(48)

12 Strategic Phase Diagram and CPUC Lobes

The CPUC channel space admits the coarse decomposition

$$\mathcal{M}_{\text{CPUC}} = L_I \cup L_X \cup L_Y \cup L_Z \cup \Sigma.$$

The equilibrium discriminant refines this decomposition:

$$\mathcal{M}_{\text{CPUC}} = \left(\bigcup_{\alpha} S_{\alpha} \right) \cup \Delta_{\text{Nash}}. \quad (49)$$

For the restricted classical-action Prisoner's Dilemma, the channel probes only the bit-flip coordinate

$$q = \frac{1 - Z}{2}.$$

The best-response differences are

$$\Delta_C(q) = A_{DC} - A_{CC} = (q - 2)(2q - 1), \quad (50)$$

$$\Delta_D(q) = A_{DD} - A_{CD} = -(q + 1)(2q - 1). \quad (51)$$

Since $0 \leq q \leq 1$, the only transition is

$$2q - 1 = 0 \iff Z = 0.$$

Hence

$$Z > 0 \implies (D, D), \quad Z < 0 \implies (C, C),$$

while $Z = 0$ is the mixed-equilibrium degeneracy set.

This example illustrates that a game may collapse a three-dimensional channel geometry onto a lower-dimensional strategic coordinate.

By contrast, the full EWL strategy space generates the nontrivial principal surface

$$Z^2 = 7XY.$$

Thus different games on the same physical channel manifold can induce different discriminants.

The four-lobe decomposition is therefore a physical coarse-graining, whereas the discriminant is game-dependent:

four-lobe CPUC geometry + game-dependent discriminant = strategic stratification.

Figure 1 illustrates this structure directly: the left panel shows the four sign-compatible lobes L_I, L_X, L_Y, L_Z of M_{CPUC} , and the right panel restricts to the $XY > 0$ sectors and overlays the discriminant surface $Z^2 = 7XY$, separating the region where (Q, Q) remains a strict best-response equilibrium from the region where it is lost. The plot shows the Q -lost region as a thin sliver near the top of the admissible cone, consistent with Eq. (46): the transition requires Z large relative to X, Y , which the CPUC inequality $|XY| \leq |Z|$ already keeps close to the boundary of M_{CPUC} .

Remark 12.1. *Each player selects a classical action $a, b \in \{C, D\}$, encoded as computational-basis bits $0 = C, 1 = D$. Before readout, each bit passes independently through the CPUC channel restricted to its classical action: by Eq. (16), the effective bit-flip probability is $q = \frac{1-Z}{2}$ (the X - and Y -Pauli components each flip the classical bit; Z -Pauli and identity leave it fixed). Writing $R_A(a', b')$ for the canonical payoff matrix of Eqs. (33)–(34) evaluated at classical bits a', b' , the expected payoff to player A playing a against B 's b is*

$$u_A(a, b; q) = \sum_{a', b' \in \{0, 1\}} P(a' | a) P(b' | b) R_A(a', b'), \quad P(a' | a) = \begin{cases} 1 - q & a' = a \\ q & a' \neq a \end{cases}.$$

Direct evaluation gives $u_A(C, C) = -q^2 - q + 3$, $u_A(D, C) = q^2 - 6q + 5$, $u_A(C, D) = q^2 + 4q$, $u_A(D, D) = -q^2 + 3q + 1$, from which

$$\Delta_C(q) = u_A(D, C) - u_A(C, C) = (q-2)(2q-1), \quad \Delta_D(q) = u_A(D, D) - u_A(C, D) = -(q+1)(2q-1),$$

matching Eqs. (51)–(52). Since $0 \leq q \leq 1$, both factors $(q-2)$ and $-(q+1)$ are sign-definite, so the only sign change of either best-response difference occurs at $2q-1=0$, i.e. $Z=0$.

13 Numerical Verification

The principal discriminant can be checked directly by evaluating

$$P_{\Delta, Q}(X, Y, Z) = Z^2 - 7XY$$

on representative physically admissible CPUC points.

For example, consider the symmetric slice

$$X = Y = r.$$

The CPUC inequalities require

$$r^2 \leq |Z|, \quad |rZ| \leq |r|,$$

and the sign condition requires $Z \geq 0$ when $r > 0$.

Take

$$(X, Y, Z) = (0.3, 0.3, 0.6).$$

Then

$$P_{\Delta,Q} = 0.6^2 - 7(0.3)(0.3) = 0.36 - 0.63 = -0.27 < 0.$$

Thus this point lies on the Q -favored side.

Take instead

$$(X, Y, Z) = (0.3, 0.3, 0.9).$$

Then

$$P_{\Delta,Q} = 0.9^2 - 7(0.3)(0.3) = 0.81 - 0.63 = 0.18 > 0.$$

This point lies on the opposite side.

The exact transition point on the same (X, Y) slice is

$$Z_* = \sqrt{7XY}.$$

For $X = Y = 0.3$,

$$Z_* = \sqrt{0.63} \approx 0.793725.$$

Both sides are physically admissible:

$$(0.3, 0.3, 0.6) \in \mathcal{M}_{\text{CPUC}}, \quad (0.3, 0.3, 0.9) \in \mathcal{M}_{\text{CPUC}},$$

and the transition point also satisfies the CPUC inequalities.

The numerical phase test is therefore

(X, Y, Z)	$P_{\Delta,Q}$	principal regime
$(0.3, 0.3, 0.6)$	-0.27	Q -favored
$(0.3, 0.3, \sqrt{0.63})$	0	discriminant
$(0.3, 0.3, 0.9)$	0.18	Q -branch lost

A complete numerical study should additionally optimize the full unilateral payoff over

$$0 \leq \theta_i \leq \pi, \quad 0 \leq \phi_i \leq \frac{\pi}{2},$$

and compare the resulting best-response values on both sides of the surface. Such a computation provides a numerical validation of the symbolic discriminant but does not replace its analytic derivation.

14 Channel-Adaptive Quantum Games

Let

$$p(t) = (X(t), Y(t), Z(t))$$

denote an estimated time-dependent channel state. A channel-adaptive player uses a policy

$$s_i(t) = \pi_i(p(t)), \quad \pi_i : \mathcal{M}_{\text{CPUC}} \rightarrow S_i.$$

The payoff becomes

$$u_i = u_i(\pi_1(p), \dots, \pi_n(p); p).$$

The discriminant provides a natural event surface for adaptive control. A controller may use the sign of

$$P_{\Delta,Q}(p(t))$$

to detect whether the system lies on one side or the other of the principal EWL strategic transition.

For a quantum network

$$G = (V, E),$$

associate to each edge e a CPUC channel

$$p_e = (X_e, Y_e, Z_e) \in \mathcal{M}_{\text{CPUC}}.$$

A path cost may then be written

$$C(P) = \sum_{e \in P} C_e(X_e, Y_e, Z_e),$$

where C_e may encode fidelity, error probability, latency, entanglement-generation rate, or memory cost.

For competing agents, this becomes a strategic routing game

$$u_i = u_i(P_1, \dots, P_n; \{p_e\}_{e \in E}).$$

The CPUC geometry supplies a continuous physical state on top of the ordinary network graph, while the equilibrium discriminant supplies a candidate partition into strategic regimes.

15 Discussion

The significance of CPUC quantum game theory is that it treats the physical quantum channel and the strategic environment as components of one mathematical object.

The map

$$p \in \mathcal{M}_{\text{CPUC}} \mapsto G_p$$

makes the channel dependence of the strategic landscape explicit. The resulting geometry can encode simultaneously:

- (i) physical channel admissibility;
- (ii) channel degeneracy;
- (iii) Pauli error structure;
- (iv) strategic parameters;
- (v) equilibrium correspondences;
- (vi) strategic transition surfaces.

The four-lobe structure supplies a discrete coarse-graining, while the continuous CPUC coordinates retain quantitative channel information.

The most concrete realization is the EWL discriminant

$$P_{\Delta, Q} = Z^2 - 7XY.$$

It demonstrates that a strategic transition need not coincide with a coordinate plane or with the physical boundary of the channel set. Instead, it can appear as a genuinely nonlinear algebraic surface in channel space.

This observation suggests a general principle:

$$\boxed{\text{quantum-channel geometry} \longrightarrow \text{equilibrium discriminant} \longrightarrow \text{strategic phase diagram.}} \quad (52)$$

The framework is therefore naturally connected to adaptive quantum networks, channel-aware routing, quantum error correction, resource allocation, and strategic quantum control. These applications are proposed interpretations of the mathematical framework rather than established performance claims.

16 Limitations

Several limitations must be stated explicitly.

First, CPUC quantum game theory is introduced here as a framework; it is not an established named theory in the quantum-game literature.

Second, the four CPUC lobes are geometric sectors and are not automatically equilibrium-invariant. Equilibrium invariance must be established for a specified game and a specified discriminant.

Third, the polynomial

$$P_{\Delta,Q} = Z^2 - 7XY$$

is a principal Q -branch best-response discriminant, not a complete elimination result for every EWL equilibrium branch.

Fourth, CPTP admissibility does not imply strategic optimality, robustness, or cryptographic security.

Fifth, a scalable quantum-network implementation requires physical realization, channel estimation, noise characterization, experimental benchmarking, and comparison with existing routing and error-correction methods.

Finally, the precise relationship between the topology of the CPUC image and the topology of complete equilibrium correspondences remains an open mathematical problem.

17 Outlook

Several directions follow directly from the present construction.

Complete EWL discriminant

The most immediate problem is to compute

$$\Delta_{\text{Nash}} = \Delta_{\text{int}} \cup \Delta_{\partial} \cup \Delta_{\text{BR}}$$

in full, including all interior, boundary, and global best-response components.

General payoff parameters

The canonical payoff tuple can be replaced by

$$T > R > P > S.$$

A symbolic elimination should determine how the coefficient 7 in

$$Z^2 - 7XY$$

changes with the payoff parameters and whether a universal normal form exists.

Asymmetric channels

The symmetric channel assumption

$$\Phi_{p,1} = \Phi_{p,2}$$

should be relaxed. The natural parameter space then becomes

$$\mathcal{M}_{\text{CPUC}} \times \mathcal{M}_{\text{CPUC}},$$

and the discriminant becomes a higher-dimensional semialgebraic variety.

Channel-adaptive quantum networks

Associate a CPUC coordinate to each network edge and define channel-aware routing and resource-allocation games. The resulting equilibrium correspondence may be studied using the same discriminant and stratification machinery.

Quantum error correction

The four CPUC sectors have a natural relation to the Pauli error classes I, X, Y, Z . This suggests a CPUC- K_4 coarse-graining for adaptive decoder selection and syndrome-processing strategies. A quantitative QEC advantage, however, requires a separate performance analysis.

Cryptography

A CPUC channel can be used as a structured physical channel model in a quantum cryptographic protocol. A complete cryptographic result would require a protocol specification and a composable security proof against a quantum adversary. The CPUC geometry alone does not provide such a security theorem.

Experimental implementation

The framework can be tested on superconducting, photonic, or other controllable qubit platforms by implementing the required Pauli/unital channel families and estimating (X, Y, Z) experimentally.

18 Conclusion

We have introduced CPUC quantum game theory as a semialgebraic geometric framework for quantum strategic behavior.

The central object is the cosine-product channel

$$(X, Y, Z) = (\cos \theta_x \cos \theta_z, \cos \theta_y \cos \theta_z, \cos \theta_x \cos \theta_y),$$

whose image is a compact semialgebraic subset of the diagonal unital qubit-channel space.

The essential mathematical construction is

$$\boxed{\mathcal{M}_{\text{CPUC}} \longrightarrow G_p \longrightarrow \mathcal{E} \longrightarrow \{S_\alpha\}.}$$

Here $\mathcal{M}_{\text{CPUC}}$ is the physical channel parameter space, G_p is the CPUC-parametrized quantum game, $\mathcal{E}$ is its equilibrium correspondence, and $\{S_\alpha\}$ is a compatible semialgebraic stratification.

The four-lobe geometry supplies a natural K_4 coarse structure, while the strategic discriminant refines the physical geometry into candidate equilibrium-invariant regimes.

For the standard maximally entangled EWL Prisoner's Dilemma, we obtained the explicit principal maximally quantum discriminant

$$\boxed{P_{\Delta, Q}(X, Y, Z) = Z^2 - 7XY.}$$

This surface provides a concrete example in which a physical three-dimensional quantum-channel manifold acquires an algebraic strategic phase boundary.

The main conceptual conclusion is therefore

$$\boxed{\text{CPUC channel geometry} \implies \text{algebraic strategic discriminant} \implies \text{strategic phase stratification}.}$$

The framework does not assert that the CPUC lobes themselves are universal equilibrium regimes, nor that the principal EWL surface is the complete Nash discriminant. Rather, it

provides a mathematically explicit procedure by which channel geometry can be converted into equilibrium geometry for specified quantum games.

This establishes a common language connecting quantum channels, semialgebraic geometry, Nash equilibrium theory, discriminants, and channel-adaptive quantum control.

References

- [1] C. Li, *The Cosine-Product Unital Channel: Semialgebraic Geometry, Stratified Topology, and Quantum Information*, ResearchGate Preprint, DOI: 10.13140/RG.2.2.25516.01925 (2026).
- [2] J. Eisert, M. Wilkens, and M. Lewenstein, “Quantum games and quantum strategies,” *Phys. Rev. Lett.* **83**, 3077–3080 (1999).
- [3] D. A. Meyer, “Quantum strategies,” *Phys. Rev. Lett.* **82**, 1052–1055 (1999).
- [4] J. Eisert and M. Wilkens, “Quantum games,” *J. Mod. Opt.* **47**, 2543–2556 (2000).
- [5] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge (2000).
- [6] M. B. Ruskai, S. Szarek, and E. Werner, “An analysis of completely-positive trace-preserving maps on 2×2 matrices,” *Linear Algebra Appl.* **347**, 159–187 (2002).
- [7] K. Siudzińska, “Geometry of symmetric and noninvertible Pauli channels,” *Phys. Rev. A* **102**, 062615 (2020).
- [8] A. Tarski, *A Decision Method for Elementary Algebra and Geometry*, University of California Press, Berkeley (1951).
- [9] S. Basu, R. Pollack, and M.-F. Roy, *Algorithms in Real Algebraic Geometry*, Springer, Berlin (2006).
- [10] Y. J. Levy, “Projections and functions of Nash equilibria,” *International Journal of Game Theory* **45**, 435–459 (2016).
- [11] D. Balkenborg and D. Vermeulen, “Semi-algebraic sets and equilibria of binary games,” *Operations Research Letters* **44**, 19–24 (2016).
- [12] Y. J. Levy, “Slicing the Nash equilibrium manifold,” *Journal of Fixed Point Theory and Applications* **25**, 85 (2023).
- [13] E. Kohlberg and J.-F. Mertens, “On the strategic stability of equilibria,” *Econometrica* **54**, 1003–1037 (1986).
- [14] J. Nash, “Non-cooperative games,” *Annals of Mathematics* **54**, 286–295 (1951).
- [15] A. Seidenberg, “A new decision method for elementary algebra,” *Annals of Mathematics* **60** (1954), 365–374.
- [16] R. M. Hardt, “Semi-algebraic local-triviality in semi-algebraic mappings,” *American Journal of Mathematics* **102** (1980), no. 2, 291–302.
- [17] J. Bochnak, M. Coste, and M.-F. Roy, *Real Algebraic Geometry*, Springer, Berlin, 1998.
- [18] L. van den Dries, *Tame Topology and O-minimal Structures*, Cambridge University Press, Cambridge, 1998.

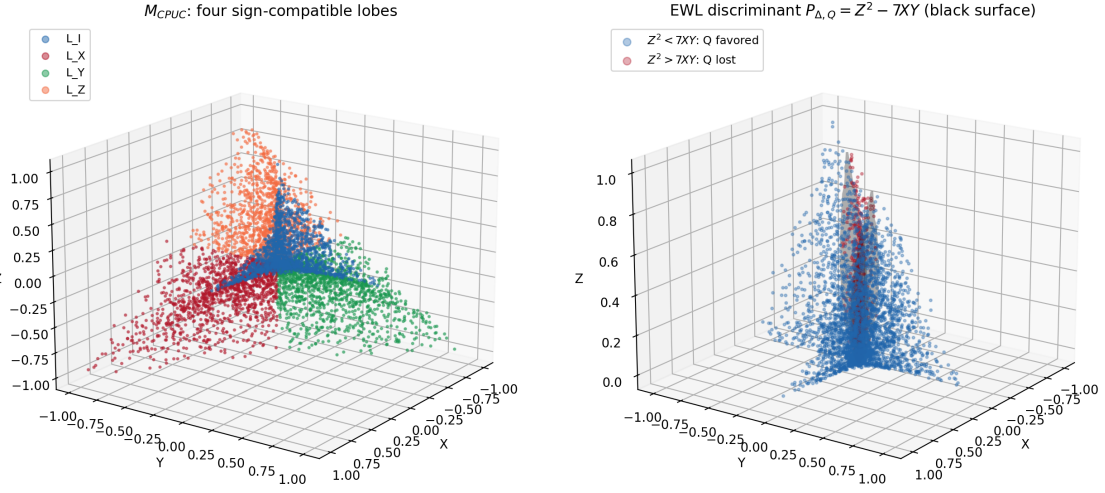


Figure 1: CPUC phase diagram. *Left*: the four sign-compatible lobes L_I, L_X, L_Y, L_Z of M_{CPUC} (Eqs. 17–20), each sampled via the cosine-product map. *Right*: restriction to the sectors with $XY > 0$, colored by the sign of the principal EWL discriminant $P_{\Delta,Q} = Z^2 - 7XY$; the black surface marks $Z^2 = 7XY$, the boundary between the Q -favored region ($Z^2 < 7XY$) and the region where (Q, Q) ceases to be a best response ($Z^2 > 7XY$).