The Cosine-Product Unital Channel (CPUC): First-Principles Quantum Dynamics, Semialgebraic Geometry, and Quantum-Information Structure

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Abstract

We derive the Cosine-Product Unital Channel (CPUC) from finite-dimensional quantum mechanics. A system qubit is coupled coherently to three ancillary qubits through the Hamiltonians

$$H_i = \frac{\omega_i}{2}\sigma_i^{(S)} \otimes \sigma_i^{(E_i)}, \qquad i \in \{x, y, z\}.$$

For initially maximally mixed ancillary states, partial trace converts the resulting system–ancilla oscillations into exact Bloch factors

$$\lambda_x = bc, \qquad \lambda_y = ac, \qquad \lambda_z = ab,$$

where

$$a = \cos\theta_x, \qquad b = \cos\theta_y, \qquad c = \cos\theta_z.$$

Thus the CPUC structure follows from coherent unitary quantum dynamics rather than from a phenomenological attenuation ansatz.

We characterize exactly the image of the parameterization

$$(a, b, c) \longmapsto (bc, ac, ab)$$

as a semialgebraic subset of the Pauli-channel parameter space. We prove that

$$XYZ \geq 0, \qquad X^2Y^2 \leq Z^2, \qquad X^2Z^2 \leq Y^2, \qquad Y^2Z^2 \leq X^2$$

are necessary and sufficient for membership in the CPUC family, together with $|X|, |Y|, |Z| \leq 1$. Complete positivity is established both by the explicit unitary dilation and by the corresponding nonnegative Pauli probabilities. We further identify the critical strata of the nonlinear parameterization and describe the resulting stratified geometry.

Contents

1 Introduction

Quantum channels provide the mathematical framework for describing the effective evolution of quantum systems under unitary dynamics, environmental coupling, measurement, and subsystem reduction. For a single qubit, an important class consists of unital channels whose action on the Bloch vector is diagonal:

$$(r_x, r_y, r_z) \longmapsto (\lambda_x r_x, \lambda_y r_y, \lambda_z r_z). \tag{1}$$

The three coefficients $(\lambda_x, \lambda_y, \lambda_z)$ provide natural coordinates for this family of channels. The physically allowed region is the Pauli-channel tetrahedron, determined by positivity of the corresponding Pauli probabilities.

The Cosine-Product Unital Channel (CPUC) [6] is the three-parameter family

$$\lambda_x = \cos\theta_y \cos\theta_z, \qquad \lambda_y = \cos\theta_x \cos\theta_z, \qquad \lambda_z = \cos\theta_x \cos\theta_y. \tag{2}$$

Introducing

$$a = \cos\theta_x, \qquad b = \cos\theta_y, \qquad c = \cos\theta_z, \tag{3}$$

the CPUC map becomes

$$\Psi : [-1,1]^3 \longrightarrow \mathbb{R}^3, \qquad \Psi(a,b,c) = (bc, ac, ab). \tag{4}$$

The purpose of this paper is to establish that this product structure has a direct quantum-dynamical origin.

The central mechanism is coherent system–ancilla interaction. A Hamiltonian of the form

$$H_i = \frac{\omega_i}{2}\sigma_i^{(S)} \otimes \sigma_i^{(E_i)} \tag{5}$$

generates finite-dimensional unitary oscillations. When the ancillary qubit is initially maximally mixed and subsequently traced out, the corresponding reduced Bloch components acquire exact cosine factors. Three coherent sectors then produce the pairwise products in (2).

This mechanism must be distinguished from purely dissipative Pauli Lindblad dynamics. A Hermitian Pauli jump operator produces exponential attenuation, whereas the CPUC cosine is generated by coherent oscillation and subsystem reduction.

The paper has four principal mathematical objectives. First, we construct an explicit finite-dimensional unitary dilation of the CPUC family. Second, we derive its Pauli-transfer and Kraus representations. Third, we characterize exactly the image of (4) by polynomial inequalities. Fourth, we analyze its singular and boundary strata.

The resulting logical chain is

$$\boxed{\begin{array}{r}\text{Hamiltonian dynamics} \longrightarrow \text{system–ancilla correlation} \longrightarrow \text{partial trace} \\ \longrightarrow \text{cosine factors} \longrightarrow \text{CPUC} \longrightarrow \text{semialgebraic geometry.}\end{array}}$$

2 First-Principles Quantum-Dynamical Construction

The CPUC construction uses standard structures and dynamical principles of finite-dimensional quantum mechanics. No phenomenological cosine attenuation law is assumed.

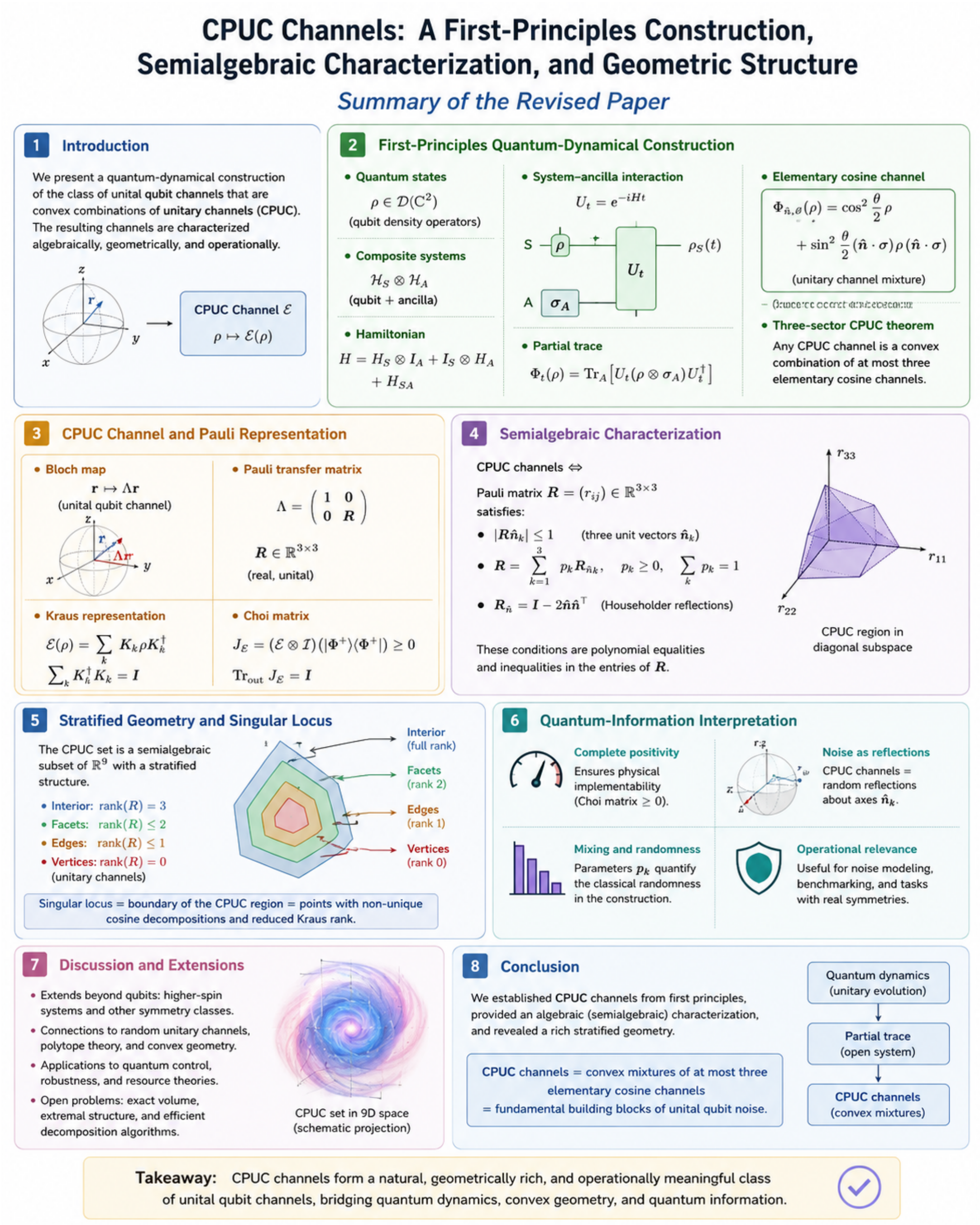

Figure 1: First-Principles of Quantum Mechanics

2.1 Quantum states

A finite-dimensional quantum system with Hilbert space $\mathcal{H}$ is described by a density operator

$$\rho \in \mathcal{B}(\mathcal{H}) \tag{6}$$

satisfying

$$\rho^\dagger = \rho, \qquad \rho \geq 0, \qquad \mathrm{Tr}\,\rho = 1. \tag{7}$$

For a qubit,

$$\mathcal{H}_S \simeq \mathbb{C}^2, \tag{8}$$

and every state admits the Bloch representation

$$\rho_S = \frac{1}{2}\left(\mathbb{I} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z\right), \tag{9}$$

where

$$\boldsymbol{r} = (r_x, r_y, r_z), \qquad \|\boldsymbol{r}\| \leq 1. \tag{10}$$

2.2 Composite systems

If S is the system and E is an ancillary environment, the composite Hilbert space is

$$\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_E. \tag{11}$$

For the CPUC realization, let

$$E = E_x \otimes E_y \otimes E_z,$$

where each E_i is a qubit. Thus

$$\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_{E_x} \otimes \mathcal{H}_{E_y} \otimes \mathcal{H}_{E_z}, \qquad \dim \mathcal{H}_{SE} = 2^4 = 16. \tag{12}$$

The ancillary system is initialized in

$$\rho_E = \frac{\mathbb{I}_{E_x}}{2} \otimes \frac{\mathbb{I}_{E_y}}{2} \otimes \frac{\mathbb{I}_{E_z}}{2} = \frac{\mathbb{I}_E}{8}. \tag{13}$$

2.3 Hamiltonian

A closed finite-dimensional quantum system evolves according to

$$\rho(t) = U(t)\rho(0)U(t)^\dagger, \qquad U(t) = e^{-iHt}, \tag{14}$$

where $H = H^\dagger$.

Equivalently,

$$\frac{d\rho}{dt} = -i[H, \rho]. \tag{15}$$

This is the Hamiltonian sector of the general Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) generator. No dissipative jump operators are required for the CPUC construction.

2.4 System–ancilla interaction

For each $i \in \{x, y, z\}$ define

$$H_i = \frac{\omega_i}{2}\sigma_i^{(S)} \otimes \sigma_i^{(E_i)}, \tag{16}$$

with the identity acting on all unused tensor factors.

Since

$$\left(\sigma_i^{(S)} \otimes \sigma_i^{(E_i)}\right)^2 = \mathbb{I}, \tag{17}$$

the propagator is

$$U_i(t) = \cos\left(\frac{\omega_i t}{2}\right)\mathbb{I} - i\sin\left(\frac{\omega_i t}{2}\right)\sigma_i^{(S)} \otimes \sigma_i^{(E_i)}. \tag{18}$$

2.5 Partial trace

The reduced state of the system is

$$\rho_S(t) = \mathrm{Tr}_E\left[\rho_{SE}(t)\right]. \tag{19}$$

For a maximally mixed ancillary qubit,

$$\rho_E = \frac{\mathbb{I}_E}{2}, \tag{20}$$

one has

$$\mathrm{Tr}_E(\sigma_\alpha \rho_E) = \frac{1}{2}\,\mathrm{Tr}(\sigma_\alpha) = 0, \qquad \alpha = x, y, z. \tag{21}$$

This property eliminates the system–environment correlation terms generated by the coherent interaction.

2.6 Elementary cosine channel

Consider the x interaction. Using

$$\{\sigma_x, \sigma_y\} = 0, \qquad \{\sigma_x, \sigma_z\} = 0,$$

direct conjugation gives

$$U_x(t)(\sigma_y^{(S)} \otimes \mathbb{I}_E)U_x(t)^\dagger = \cos(\omega_x t)\sigma_y^{(S)} \otimes \mathbb{I}_E + \sin(\omega_x t)\sigma_z^{(S)} \otimes \sigma_x^{(E_x)}, \tag{22}$$

$$U_x(t)(\sigma_z^{(S)} \otimes \mathbb{I}_E)U_x(t)^\dagger = \cos(\omega_x t)\sigma_z^{(S)} \otimes \mathbb{I}_E - \sin(\omega_x t)\sigma_y^{(S)} \otimes \sigma_x^{(E_x)}, \tag{23}$$

$$U_x(t)(\sigma_x^{(S)} \otimes \mathbb{I}_E)U_x(t)^\dagger = \sigma_x^{(S)} \otimes \mathbb{I}_E. \tag{24}$$

After tracing out E_x,

$$\sigma_x \mapsto \sigma_x, \qquad \sigma_y \mapsto \cos(\omega_x t)\sigma_y, \qquad \sigma_z \mapsto \cos(\omega_x t)\sigma_z. \tag{25}$$

Therefore

$$\Phi_x : (r_x, r_y, r_z) \longmapsto (r_x, c_x r_y, c_x r_z), \qquad c_x = \cos\theta_x. \tag{26}$$

Cyclic permutation gives

$$\Phi_y : (r_x, r_y, r_z) \longmapsto (c_y r_x, r_y, c_y r_z), \tag{27}$$

$$\Phi_z : (r_x, r_y, r_z) \longmapsto (c_z r_x, c_z r_y, r_z). \tag{28}$$

2.7 Three-sector CPUC theorem

Let

$$a = \cos\theta_x, \qquad b = \cos\theta_y, \qquad c = \cos\theta_z. \tag{29}$$

Sequential composition gives

$$\Phi_{\mathrm{CPUC}} = \Phi_z \circ \Phi_y \circ \Phi_x, \tag{30}$$

and hence

$$(r_x, r_y, r_z) \longmapsto (bc\, r_x, ac\, r_y, ab\, r_z). \tag{31}$$

Theorem 2.1 (Finite-dimensional unitary realization of CPUC)**.** *For every*

$$(a, b, c) \in [-1, 1]^3,$$

there exists a finite-dimensional system–ancilla unitary evolution, with one system qubit and three ancillary qubits initially in the maximally mixed state, whose reduced channel is

$$\Phi_{a,b,c} : (r_x, r_y, r_z) \longmapsto (bc\, r_x, ac\, r_y, ab\, r_z). \tag{32}$$

Proof. Choose

$$\theta_x, \theta_y, \theta_z \in [0, \pi]$$

such that

$$a = \cos\theta_x, \qquad b = \cos\theta_y, \qquad c = \cos\theta_z.$$

Set

$$H_i = \frac{\theta_i}{2t_i}\sigma_i^{(S)} \otimes \sigma_i^{(E_i)}.$$

The elementary reductions are given by (26)–(28). Their composition therefore yields

$$(\lambda_x, \lambda_y, \lambda_z) = (bc, ac, ab).$$

$\square$

3 CPUC Channel and Pauli Representation

3.1 Bloch map

Define

$$X = bc, \qquad Y = ac, \qquad Z = ab. \tag{33}$$

The CPUC channel acts as

$$\boxed{(r_x, r_y, r_z) \longmapsto (Xr_x, Yr_y, Zr_z).} \tag{34}$$

Equivalently,

$$\Phi_{\mathrm{CPUC}}(\rho) = \frac{1}{2}\left[\mathbb{I} + Xr_x\sigma_x + Yr_y\sigma_y + Zr_z\sigma_z\right]. \tag{35}$$

3.2 Pauli transfer matrix

Using the ordered Pauli basis

$$\{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\},$$

the Pauli transfer matrix is

$$T_{\mathrm{CPUC}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & 0 & Y & 0 \\ 0 & 0 & 0 & Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & bc & 0 & 0 \\ 0 & 0 & ac & 0 \\ 0 & 0 & 0 & ab \end{pmatrix}. \tag{36}$$

The elementary transfer matrices satisfy

$$T_{\mathrm{CPUC}} = T_z T_y T_x. \tag{37}$$

3.3 Kraus representation

Every diagonal unital qubit Pauli channel has the form

$$\Phi(\rho) = \sum_{\mu=0,x,y,z} p_\mu \sigma_\mu \rho \sigma_\mu, \qquad \sigma_0 = \mathbb{I}. \tag{38}$$

For the CPUC channel,

$$p_0 = \frac{1+X+Y+Z}{4} = \frac{1+bc+ac+ab}{4}, \tag{39}$$
$$p_x = \frac{1+X-Y-Z}{4} = \frac{1+bc-ac-ab}{4}, \tag{40}$$
$$p_y = \frac{1-X+Y-Z}{4} = \frac{1-bc+ac-ab}{4}, \tag{41}$$
$$p_z = \frac{1-X-Y+Z}{4} = \frac{1-bc-ac+ab}{4}. \tag{42}$$

A Kraus representation is therefore

$$K_0 = \sqrt{p_0}\mathbb{I}, \qquad K_x = \sqrt{p_x}\sigma_x, \qquad K_y = \sqrt{p_y}\sigma_y, \qquad K_z = \sqrt{p_z}\sigma_z. \tag{43}$$

The normalization condition is

$$\sum_\mu K_\mu^\dagger K_\mu = \mathbb{I}. \tag{44}$$

For $a, b, c \in [-1, 1]$, the coefficients admit the factorizations

$$p_0 = \frac{1}{8}\left[(1+a)(1+b)(1+c) + (1-a)(1-b)(1-c)\right], \tag{45}$$
$$p_x = \frac{1}{8}\left[(1+a)(1-b)(1-c) + (1-a)(1+b)(1+c)\right], \tag{46}$$
$$p_y = \frac{1}{8}\left[(1+a)(1-b)(1+c) + (1-a)(1+b)(1-c)\right], \tag{47}$$
$$p_z = \frac{1}{8}\left[(1+a)(1+b)(1-c) + (1-a)(1-b)(1+c)\right]. \tag{48}$$

Thus

$$p_0, p_x, p_y, p_z \geq 0. \tag{49}$$

3.4 Choi matrix

Let

$$J(\Phi) = (\Phi \otimes \mathbb{I})(|\Omega\rangle\langle\Omega|), \qquad |\Omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{50}$$

For the CPUC unital channel,

$$J(\Phi_{\text{CPUC}}) = p_0 \left|\Phi^+\right\rangle\left\langle\Phi^+\right| + p_x \left|\Psi^+\right\rangle\left\langle\Psi^+\right| + p_y \left|\Psi^-\right\rangle\left\langle\Psi^-\right| + p_z \left|\Phi^-\right\rangle\left\langle\Phi^-\right|, \tag{51}$$

up to the conventional assignment of Bell states to Pauli operators.

Consequently,

$$\operatorname{spec} J(\Phi_{\text{CPUC}}) = \{p_0, p_x, p_y, p_z\}. \tag{52}$$

Therefore

$$\Phi_{\text{CPUC}} \text{ is CP} \iff J(\Phi_{\text{CPUC}}) \geq 0 \iff p_\mu \geq 0. \tag{53}$$

The unitary dilation independently establishes complete positivity:

$$\Phi_{a,b,c}(\rho) = \operatorname{Tr}_E\left[U\left(\rho \otimes \frac{\mathbb{I}_E}{8}\right)U^\dagger\right]. \tag{54}$$

Since adjoining a state, unitary conjugation, and partial trace are all CPTP operations,

$$\boxed{\Phi_{a,b,c} \text{ is CPTP.}} \tag{55}$$

Thus complete positivity follows both from quantum dilation and from the explicit Choi spectrum.

4 Semialgebraic Characterization

Define

$$\Psi(a,b,c) = (bc, ac, ab) \tag{56}$$

and

$$\mathcal{M} = \Psi([-1,1]^3). \tag{57}$$

The set $\mathcal{M}$ is a compact semialgebraic subset of $\mathbb{R}^3$.

4.1 Necessary conditions

If

$$(X,Y,Z) = (bc, ac, ab),$$

then

$$XYZ = (bc)(ac)(ab) = a^2b^2c^2 \geq 0. \tag{58}$$

Moreover,

$$X^2Y^2 = a^2b^2c^4 \leq a^2b^2 = Z^2, \tag{59}$$
$$X^2Z^2 = a^2b^4c^2 \leq a^2c^2 = Y^2, \tag{60}$$
$$Y^2Z^2 = a^4b^2c^2 \leq b^2c^2 = X^2. \tag{61}$$

Hence

$$X^2Y^2 \leq Z^2, \qquad X^2Z^2 \leq Y^2, \qquad Y^2Z^2 \leq X^2. \tag{62}$$

4.2 Sufficiency

Theorem 4.1 (Semialgebraic characterization)**.** *A point $(X,Y,Z) \in [-1,1]^3$ belongs to $\mathcal{M}$ if and only if*

$$XYZ \geq 0, \tag{63}$$

and

$$X^2Y^2 \leq Z^2, \qquad X^2Z^2 \leq Y^2, \qquad Y^2Z^2 \leq X^2. \tag{64}$$

Proof. The necessity was proved above.

Suppose first that $XYZ \neq 0$. Define

$$a = \operatorname{sgn}(YZ)\sqrt{\frac{YZ}{X}}, \qquad b = \operatorname{sgn}(XZ)\sqrt{\frac{XZ}{Y}}, \qquad c = \operatorname{sgn}(XY)\sqrt{\frac{XY}{Z}}, \tag{65}$$

where the square roots are real because $XYZ \geq 0$.

The polynomial inequalities imply

$$|a|, |b|, |c| \leq 1.$$

Direct substitution gives

$$bc = X, \qquad ac = Y, \qquad ab = Z.$$

Thus $(X,Y,Z) \in \mathcal{M}$.

If one or more coordinates vanish, the same conclusion follows by choosing the nonzero coordinates appropriately. In particular, if $X = Y = Z = 0$, one may choose

$$(a,b,c) = (0,0,0),$$

although this preimage is not unique. □

4.3 CPTP characterization

Combining Theorem 4.1 with the Pauli probabilities gives

$$\boxed{\mathcal{M}=\left\{(X,Y,Z)\in[-1,1]^3:\begin{array}{c}XYZ\geq 0,\\ X^2Y^2\leq Z^2,\\ X^2Z^2\leq Y^2,\\ Y^2Z^2\leq X^2\end{array}\right\}.} \tag{66}$$

Every point of $\mathcal{M}$ therefore defines a unital CPTP Pauli channel with eigenvalues (X,Y,Z).

5 Stratified Geometry and Singular Locus

The nonlinear map

$$\Psi(a,b,c)=(bc,ac,ab) \tag{67}$$

contains the essential geometry of the CPUC family.

Globally, $\mathcal{M}$ should be regarded as a compact semialgebraic stratified set rather than as a smooth manifold.

5.1 Jacobian and regular locus

The Jacobian matrix is

$$D\Psi=\begin{pmatrix}0&c&b\\c&0&a\\b&a&0\end{pmatrix}. \tag{68}$$

Its determinant is

$$\det D\Psi=2abc. \tag{69}$$

Hence

$$\operatorname{rank} D\Psi=3 \qquad \text{whenever } abc\neq 0. \tag{70}$$

The regular parameter locus is therefore

$$\mathcal{R}=\{(a,b,c)\in(-1,1)^3: abc\neq 0\}. \tag{71}$$

The critical parameter locus is

$$\mathcal{C}=\{abc=0\}. \tag{72}$$

5.2 Critical-value locus

The coordinate planes map to coordinate axes:

$$a=0\Longrightarrow(X,Y,Z)=(bc,0,0), \tag{73}$$

$$b=0\Longrightarrow(X,Y,Z)=(0,ac,0), \tag{74}$$

$$c=0\Longrightarrow(X,Y,Z)=(0,0,ab). \tag{75}$$

Therefore

Proposition 5.1 (Critical-value locus)**.** *The critical-value locus of Ψ is*

$$\operatorname{CritVal}(\Psi)=\{(X,0,0):|X|\leq 1\}\cup\{(0,Y,0):|Y|\leq 1\}\cup\{(0,0,Z):|Z|\leq 1\}. \tag{76}$$

These coordinate axes constitute distinguished lower-dimensional strata of the CPUC semialgebraic set.

5.3 The origin and its non-unique preimage

The origin

$$(X, Y, Z) = (0, 0, 0)$$

corresponds to the completely depolarizing channel

$$\Phi_0(\rho) = \frac{\mathbb{I}}{2}. \tag{77}$$

Importantly, the preimage of the origin is not a single point. Indeed,

$$bc = ac = ab = 0$$

if and only if at least two of a, b, c vanish. Hence

$$\Psi^{-1}(0, 0, 0) = \{(a, 0, 0) : a \in [-1, 1]\} \cup \{(0, b, 0) : b \in [-1, 1]\} \cup \{(0, 0, c) : c \in [-1, 1]\}. \tag{78}$$

Thus $(a, b, c) = (0, 0, 0)$ is only one representative of the full preimage.

5.4 Local quadratic structure

Near any preimage of the origin, the parameterization retains its exact quadratic form

$$X = bc, \qquad Y = ac, \qquad Z = ab. \tag{79}$$

Writing

$$a = u, \qquad b = v, \qquad c = w,$$

gives

$$q(u, v, w) = (vw, uw, uv). \tag{80}$$

The algebraic cone associated with this quadratic map is

$$\mathcal{C}_0 = \{(vw, uw, uv) : (u, v, w) \in \mathbb{R}^3\}. \tag{81}$$

Because

$$q(tu, tv, tw) = t^2 q(u, v, w),$$

the set $\mathcal{C}_0$ is a genuine cone. Its nontrivial local structure is therefore controlled by the quadratic factorization itself.

A complete topological classification of the link of this cone is not needed for the quantum-dynamical realization and is not asserted here.

5.5 Boundary strata

The boundary of the parameter cube

$$[-1, 1]^3$$

corresponds to one or more coherent amplitudes satisfying

$$|a| = 1, \qquad |b| = 1, \qquad |c| = 1.$$

Since

$$a = \cos\theta_x,$$

the condition $|a| = 1$ corresponds to

$$\theta_x \in \pi\mathbb{Z},$$

and similarly for b, c.

These strata correspond to coherent interactions whose associated cosine factor has unit magnitude and hence whose corresponding elementary reduced channel leaves two Bloch components unchanged up to sign.

6 Quantum-Information Interpretation

6.1 Pairwise environmental factors

The CPUC eigenvalues are

$$\lambda_x = bc, \qquad \lambda_y = ac, \qquad \lambda_z = ab. \tag{82}$$

Each Bloch direction therefore receives two elementary coherent factors.

The structure can be represented by three microscopic amplitudes

$$a,\ b,\ c$$

and three effective pairwise products

$$ab,\ ac,\ bc.$$

Thus the CPUC map may be interpreted as an edge-product map from a three-factor parameter space to a three-dimensional channel space.

6.2 Coherent attenuation versus dissipative decay

The CPUC cosine factors should not be identified with ordinary irreversible exponential decoherence.

A Hermitian Pauli jump

$$L = \sqrt{\gamma}\sigma_x \tag{83}$$

gives the GKSL equation

$$\frac{d\rho}{dt} = \gamma\left(\sigma_x\rho\sigma_x - \rho\right), \tag{84}$$

which produces

$$r_y(t) = e^{-2\gamma t} r_y(0), \qquad r_z(t) = e^{-2\gamma t} r_z(0). \tag{85}$$

By contrast, the CPUC construction produces

$$r_y(t) = \cos(\omega_x t) r_y(0), \qquad r_z(t) = \cos(\omega_x t) r_z(0). \tag{86}$$

Thus

$$\boxed{\text{coherent information transfer} \longrightarrow \text{system–environment correlation} \longrightarrow \text{partial trace} \longrightarrow \text{effective cosine atten}} \tag{87}$$

6.3 Dilation interpretation

The CPUC unital channel can be written as

$$\Phi_{a,b,c}(\rho) = \mathrm{Tr}_E\left[U_{a,b,c}\left(\rho\otimes\frac{\mathbb{I}_E}{8}\right)U_{a,b,c}^\dagger\right], \tag{88}$$

where

$$U_{a,b,c} = U_z U_y U_x \tag{89}$$

and

$$U_i = \exp\left[-i\frac{\theta_i}{2}\sigma_i^{(S)}\otimes\sigma_i^{(E_i)}\right]. \tag{90}$$

This is a finite-dimensional unitary dilation of the CPUC unital channel.

The enlarged dynamics is generated by Hamiltonian GKSL generators. The reduced channel, however, need not form a time-homogeneous Markov semigroup. This distinction is essential when interpreting the oscillatory cosine coefficients.

6.4 CPTP structure

The dilation immediately implies

$$\Phi_{a,b,c} \in \mathrm{CPTP}. \tag{91}$$

The explicit Pauli representation provides an independent characterization:

$$\Phi_{a,b,c} = p_0\mathcal{I} + p_x\mathcal{X} + p_y\mathcal{Y} + p_z\mathcal{Z}, \tag{92}$$

where

$$\mathcal{X}(\rho) = \sigma_x \rho \sigma_x,$$

and similarly for $\mathcal{Y}, \mathcal{Z}$.

Hence the CPUC family lies inside the tetrahedral region of unital qubit Pauli channels, with its own nonlinear semialgebraic constraints inherited from the factorization

$$(X, Y, Z) = (bc, ac, ab).$$

6.5 Maximal contraction

At

$$X = Y = Z = 0,$$

the channel is completely depolarizing:

$$\Phi_0(\rho) = \frac{\mathbb{I}}{2}. \tag{93}$$

The corresponding channel point is highly non-injective under the CPUC parameterization, as shown by (78). Thus the maximally contractive channel is simultaneously a distinguished quantum information point and a singular value of the nonlinear parameter map.

7 Discussion and Extensions

The CPUC construction establishes a direct bridge between microscopic quantum dynamics and nonlinear geometry in channel space.

The essential mechanism is

$$\boxed{\begin{aligned} &\text{finite-dimensional Hamiltonian} \to \text{unitary oscillation} \\ &\qquad \to \text{system–ancilla correlation} \to \text{partial trace} \\ &\qquad \to \text{pairwise cosine products.} \end{aligned}} \tag{94}$$

The parameters a, b, c have a natural physical interpretation as coherent interaction amplitudes,

$$a = \cos\theta_x, \qquad b = \cos\theta_y, \qquad c = \cos\theta_z,$$

whereas X, Y, Z are effective reduced-channel coordinates.

The nonlinear relation

$$(X, Y, Z) = (bc, ac, ab)$$

is therefore not merely an algebraic parametrization. It records the effect of eliminating ancillary quantum degrees of freedom.

Several extensions are natural.

First, one may replace the ancillary qubits by higher-dimensional environments and investigate which generalized product structures arise.

Second, one may add genuine dissipative GKSL terms to the coherent Hamiltonian:

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_{\alpha} \gamma_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\} \right). \tag{95}$$

Such extensions can generate damped oscillatory structures of the schematic form

$$\lambda_x(t) = e^{-\Gamma_x t} \cos\theta_y(t) \cos\theta_z(t), \tag{96}$$

and similarly for the other Bloch directions.

Third, the three-ancilla realization may be interpreted as a minimal finite-dimensional dilation adapted to the pairwise factorization. It would be interesting to determine whether lower-dimensional environments can realize the complete CPUC family without additional controls.

Fourth, the semialgebraic geometry suggests a broader class of channel families generated by monomial maps

$$\lambda_i = \prod_j a_j^{m_{ij}}, \tag{97}$$

with coherent amplitudes $a_j \in [-1,1]$. The CPUC family corresponds to the exponent matrix

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{98}$$

The corresponding image problem naturally belongs to semialgebraic and toric-type geometry.

Finally, the channel geometry may be connected to applications in quantum games. In particular, when the CPUC channel is inserted into a quantum game protocol, the variables (X,Y,Z) provide physically constrained coordinates for the resulting payoff landscape. Any discriminant or phase diagram derived from such a game should therefore be interpreted inside the CPUC semialgebraic set rather than in the full unconstrained cube $[-1,1]^3$.

8 Conclusion

We have derived the Cosine-Product Unital Channel from a concrete finite-dimensional quantum-dynamical construction.

A system qubit is coupled coherently to three ancillary qubits through

$$H_i = \frac{\omega_i}{2} \sigma_i^{(S)} \otimes \sigma_i^{(E_i)}.$$

Unitary evolution produces exact trigonometric oscillations. Because the ancillas are initially maximally mixed, the system–environment correlation terms vanish under the partial trace, leaving cosine coefficients on two Bloch directions for each elementary interaction.

Sequential composition therefore gives

$$\boxed{(\lambda_x, \lambda_y, \lambda_z) = (bc, ac, ab).}$$

This establishes the CPUC family as a genuine reduced quantum channel arising from finite-dimensional unitary dynamics.

The same family admits the Pauli-channel representation

$$\Phi(\rho) = \sum_{\mu=0,x,y,z} p_\mu \sigma_\mu \rho \sigma_\mu,$$

with explicitly nonnegative coefficients p_μ. Equivalently, its Choi matrix is positive semidefinite. Complete positivity and trace preservation therefore follow both from the quantum dilation and from the explicit Pauli representation.

The image of the nonlinear map

$$\Psi(a, b, c) = (bc, ac, ab)$$

is the compact semialgebraic set

$$\mathcal{M} = \left\{ (X, Y, Z) \in [-1, 1]^3 : \begin{array}{r} XYZ \geq 0, \\ X^2Y^2 \leq Z^2, \\ X^2Z^2 \leq Y^2, \\ Y^2Z^2 \leq X^2 \end{array} \right\}.$$

The Jacobian determinant

$$\det D\Psi = 2abc$$

identifies the coordinate planes as the critical strata of the parameterization. Their images are the three coordinate axes in channel space. The origin corresponds to the completely depolarizing channel and possesses a nontrivial preimage, reflecting the singular nature of the parameterization.

The principal structural result can therefore be summarized as

$$\boxed{\begin{array}{r} \text{finite-dimensional quantum mechanics} \longrightarrow \text{unitary system–ancilla dynamics} \\ \longrightarrow \text{cosine-product factorization} \longrightarrow \text{CPUC semialgebraic set} \end{array}}$$

The CPUC family consequently provides a concrete example in which a nonlinear semialgebraic structure in quantum-channel space has a direct microscopic interpretation in terms of coherent quantum dynamics and environmental reduction.

A Gate-Level Quantum-Circuit Realization of the CPUC Dilation

This appendix gives an explicit gate-level realization of the finite-dimensional unitary dilation introduced in Section 2. The construction shows that the cosine-product unital channel (CPUC) can be implemented using only standard one- and two-qubit quantum gates, followed by discarding the ancillary degrees of freedom. In particular, no non-Hermitian dissipator is required at the level of the enlarged closed quantum system.

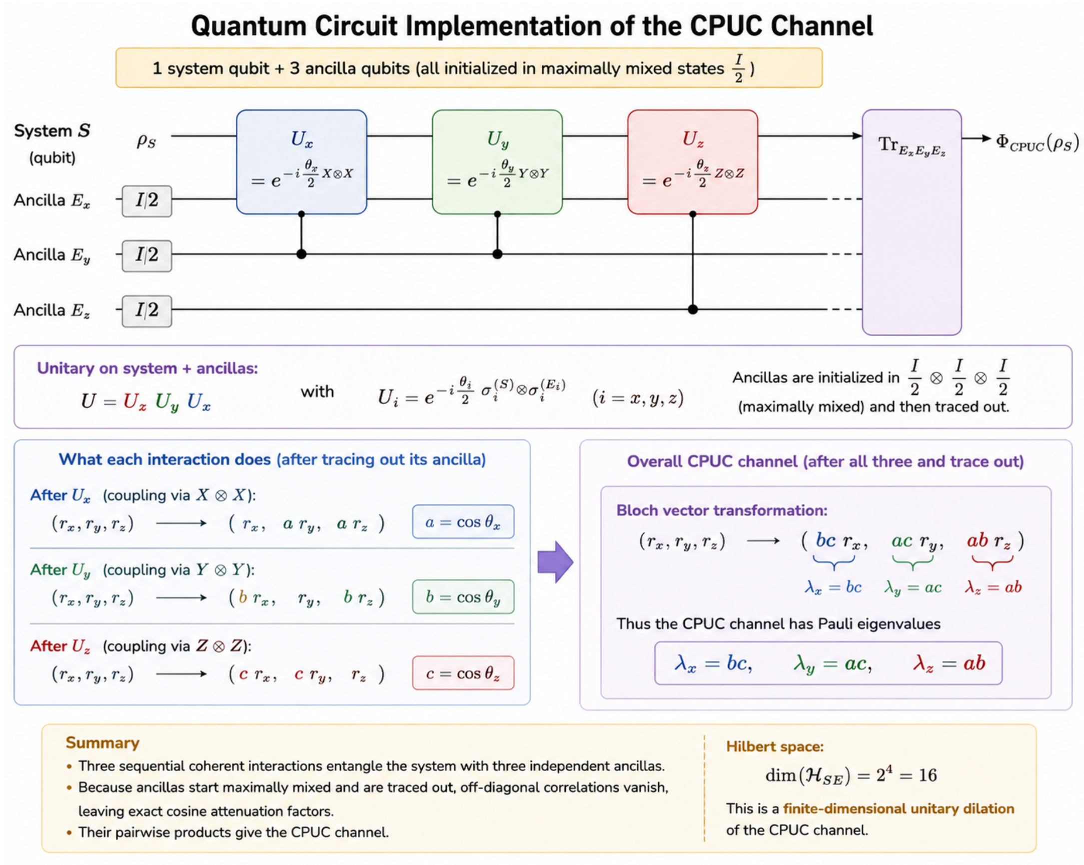

Figure 2: Quantum Circuit of CPUC Dilation

Throughout this appendix, the system qubit is denoted by S, while E_x, E_y, E_z denote three ancillary qubits. We write

$$a = \cos\theta_x, \qquad b = \cos\theta_y, \qquad c = \cos\theta_z. \tag{99}$$

The target CPUC channel is

$$\Phi_{\mathrm{CPUC}} : (r_x, r_y, r_z) \longmapsto (bc\,r_x,\ ac\,r_y,\ ab\,r_z). \tag{100}$$

The Hermitian generates ancilla interactionsem–ancilla interactions are generated by the Hermitian Hamiltonians

$$H_x = \frac{\omega_x}{2} X_S \otimes X_{E_x}, \qquad H_y = \frac{\omega_y}{2} Y_S \otimes Y_{E_y}, \qquad H_z = \frac{\omega_z}{2} Z_S \otimes Z_{E_z}. \tag{101}$$

For an interaction time t, define

$$\theta_i = \omega_i t, \qquad i \in \{x, y, z\}. \tag{102}$$

The corresponding unitary operators are

$$U_x = \exp\left(-i\frac{\theta_x}{2} X_S X_{E_x}\right), \tag{103}$$

$$U_y = \exp\left(-i\frac{\theta_y}{2} Y_S Y_{E_y}\right), \tag{104}$$

and

$$U_z = \exp\left(-i\frac{\theta_z}{2} Z_S Z_{E_z}\right). \tag{105}$$

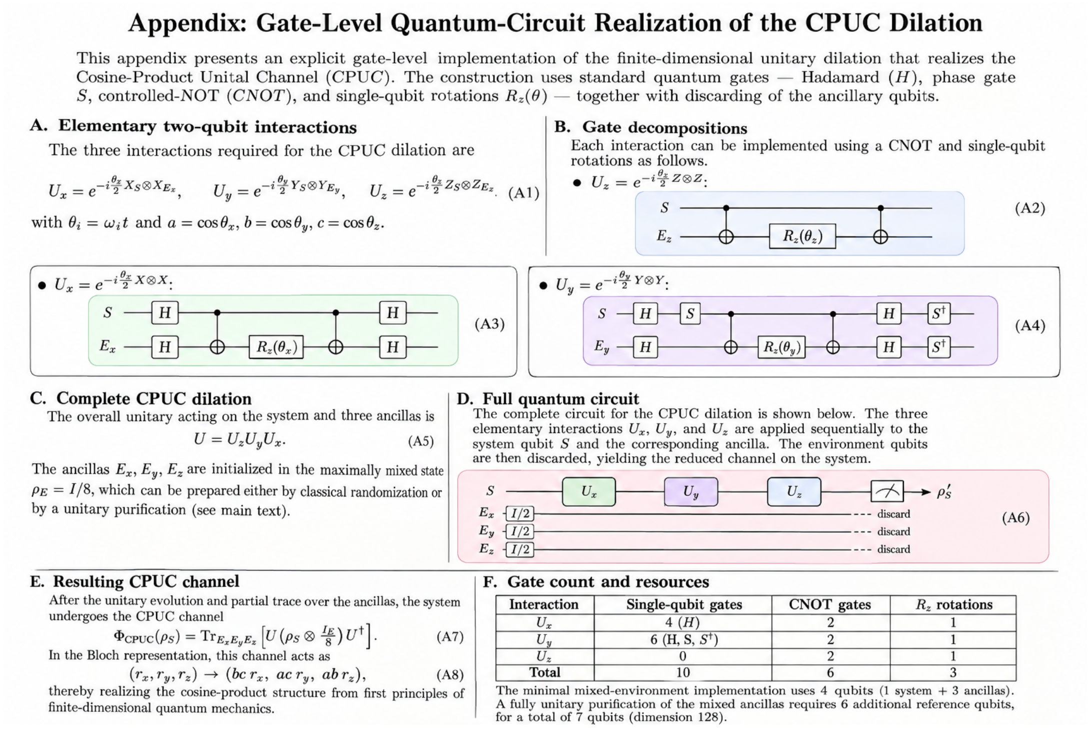

Figure 3: Quantum Circuit of CPUC Dilation

For any Pauli operator $P \in \{X, Y, Z\}$, a two-qubit Pauli rotation

$$R_{PP}(\theta) = \exp\left(-i\frac{\theta}{2}P\otimes P\right) \tag{106}$$

can be reduced to a ZZ rotation by a local Clifford basis transformation.

The fundamental ZZ identity is

$$\exp\left(-i\frac{\theta}{2}Z\otimes Z\right) = \mathrm{CNOT}\,(I\otimes R_z(\theta))\,\mathrm{CNOT}, \tag{107}$$

where

$$R_z(\theta) = \exp\left(-i\frac{\theta}{2}Z\right). \tag{108}$$

A.1 Gate decomposition of the XX interaction

Since

$$HZH = X, \tag{109}$$

the XX interaction is obtained from the ZZ interaction by conjugation with Hadamard gates:

$$U_x = (H_S\otimes H_{E_x})\,\mathrm{CNOT}_{S,E_x}\,(I_S\otimes R_z(\theta_x))\,\mathrm{CNOT}_{S,E_x}(H_S\otimes H_{E_x}). \tag{110}$$

Equivalently, the gate sequence, read from left to right, is

$$H_S,\ H_{E_x},\ \mathrm{CNOT}_{S,E_x},\ R_z^{(E_x)}(\theta_x),\ \mathrm{CNOT}_{S,E_x},\ H_S,\ H_{E_x}. \tag{111}$$

The corresponding circuit is

A.2 Gate decomposition of the YY interaction

For the YY interaction, define the local Clifford transformation

$$B_Y = SH. \tag{112}$$

Since

$$B_Y Z B_Y^\dagger = SHZHS^\dagger = SXS^\dagger = Y, \tag{113}$$

the YY rotation is obtained by conjugating the ZZ rotation:

$$U_y = (B_Y \otimes B_Y)\,\mathrm{CNOT}_{S,E_y}\,(I_S \otimes R_z(\theta_y))\,\mathrm{CNOT}_{S,E_y}\,(B_Y^\dagger \otimes B_Y^\dagger). \tag{114}$$

Because

$$B_Y^\dagger = HS^\dagger, \tag{115}$$

the gate sequence is

$$H_S,\ S_S,\ H_{E_y},\ S_{E_y},\ \mathrm{CNOT}_{S,E_y},\ R_z^{(E_y)}(\theta_y),\ \mathrm{CNOT}_{S,E_y},\ H_S,\ S_S^\dagger,\ H_{E_y},\ S_{E_y}^\dagger. \tag{116}$$

Thus the circuit realizes exactly

$$U_y = \exp\left(-i\frac{\theta_y}{2} Y_S Y_{E_y}\right). \tag{117}$$

A.3 Gate decomposition of the ZZ interaction

The ZZ interaction requires no basis transformation:

$$U_z = \mathrm{CNOT}_{S,E_z}\,(I_S \otimes R_z(\theta_z))\,\mathrm{CNOT}_{S,E_z}\,. \tag{118}$$

and therefore implements

$$U_z = \exp\left(-i\frac{\theta_z}{2} Z_S Z_{E_z}\right). \tag{119}$$

A.4 Preparation of the ancillary states

The reduced CPUC construction uses the initial ancillary state

$$\rho_E = \frac{I_{E_x}}{2} \otimes \frac{I_{E_y}}{2} \otimes \frac{I_{E_z}}{2} = \frac{I_E}{8}. \tag{120}$$

A maximally mixed qubit may be prepared experimentally by classical randomization. Starting from $|0\rangle$, apply X with probability 1/2 and I with probability 1/2. The resulting density operator is

$$\rho_E = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{I}{2}. \tag{121}$$

For a completely unitary implementation, the maximally mixed state can instead be purified. Introduce a reference qubit R_i for each ancilla E_i and prepare the Bell state

$$|\Phi^+\rangle_{E_i R_i} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{122}$$

The reduced state satisfies

$$\mathrm{Tr}_{R_i}\left[|\Phi^+\rangle\langle\Phi^+|\right] = \frac{I_{E_i}}{2}. \tag{123}$$

The Bell state is prepared using

$$|00\rangle \xrightarrow{H\otimes I} \xrightarrow{\mathrm{CNOT}} |\Phi^+\rangle. \tag{124}$$

Consequently, the maximally mixed environment can be realized as the reduced state of a larger pure environment. This gives a fully unitary purification of the CPUC dilation.

A.5 Complete CPUC circuit

The three elementary interactions are applied sequentially:

$$U_{\text{CPUC}} = U_z U_y U_x. \tag{125}$$

The total system–environment evolution is therefore

$$\rho'_{SE} = U_{\text{CPUC}} \left(\rho_S \otimes \frac{I_E}{8} \right) U^\dagger_{\text{CPUC}}. \tag{126}$$

The reduced system state is

$$\boxed{\Phi_{\text{CPUC}}(\rho_S) = \text{Tr}_{E_x E_y E_z} \left[U_z U_y U_x \left(\rho_S \otimes \frac{I_E}{8} \right) U^\dagger_x U^\dagger_y U^\dagger_z \right].} \tag{127}$$

A.6 Partial trace as an operational discard

The partial trace is not a unitary gate and does not require a physical gate operation. It is the mathematical description of ignoring the environmental subsystem. For a bipartite state

$$\rho_{SE} = \sum_{i,j,k,l} \rho_{ij,kl} |i\rangle\langle j|_S \otimes |k\rangle\langle l|_E, \tag{128}$$

the reduced system state is

$$\text{Tr}_E(\rho_{SE}) = \sum_{i,j,k} \rho_{ij,kk} |i\rangle\langle j|_S. \tag{129}$$

Experimentally, measuring only the system qubit while retaining no information about the environment produces precisely the statistics associated with this reduced state. Hence

$$\boxed{\text{Tr}_E \quad \longleftrightarrow \quad \text{discarding or ignoring the environment.}} \tag{130}$$

A.7 Recovery of the elementary cosine channels

The gate-level construction reproduces the elementary channels used in the CPUC theorem. For the x interaction,

$$\Phi_x(\rho_S) = \text{Tr}_{E_x} \left[U_x \left(\rho_S \otimes \frac{I}{2} \right) U^\dagger_x \right]. \tag{131}$$

Its action on the Pauli basis is

$$\Phi_x(I) = I, \qquad \Phi_x(X) = X, \qquad \Phi_x(Y) = aY, \qquad \Phi_x(Z) = aZ. \tag{132}$$

Therefore

$$(r_x, r_y, r_z) \xmapsto{\Phi_x} (r_x, ar_y, ar_z). \tag{133}$$

Similarly,

$$(r_x, r_y, r_z) \xmapsto{\Phi_y} (br_x, r_y, br_z), \tag{134}$$

and

$$(r_x, r_y, r_z) \xmapsto{\Phi_z} (cr_x, cr_y, r_z). \tag{135}$$

The composition gives

$$\Phi_{\text{CPUC}} = \Phi_z \circ \Phi_y \circ \Phi_x, \tag{136}$$

$$(r_x, r_y, r_z) \longmapsto (bc\, r_x,\ ac\, r_y,\ ab\, r_z). \tag{137}$$

Thus the cosine-product eigenvalues follow directly from coherent system–ancilla Hamiltonian evolution:

$$\boxed{\lambda_x = bc, \qquad \lambda_y = ac, \qquad \lambda_z = ab.} \tag{138}$$

A.8 Kraus representation induced by the circuit

The same circuit immediately yields a Kraus representation. For a single maximally mixed ancillary qubit,

$$\Phi_i(\rho) = \frac{1}{2}\sum_{\mu=0}^{1} \mathrm{Tr}_{E_i}\left[U_i\left(\rho \otimes |\mu\rangle\langle\mu|\right)U_i^\dagger\right]. \tag{139}$$

Equivalently, the Kraus operators may be taken as

$$K_{\alpha\mu}^{(i)} = \frac{1}{\sqrt{2}}\langle\alpha|U_i|\mu\rangle, \qquad \alpha, \mu \in \{0, 1\}, \tag{140}$$

so that

$$\Phi_i(\rho) = \sum_{\alpha,\mu} K_{\alpha\mu}^{(i)} \rho K_{\alpha\mu}^{(i)\dagger}. \tag{141}$$

The completeness relation

$$\sum_{\alpha,\mu} K_{\alpha\mu}^{(i)\dagger} K_{\alpha\mu}^{(i)} = I \tag{142}$$

follows from the unitarity of U_i.

Consequently, complete positivity and trace preservation of the elementary cosine channels are not imposed independently at the gate level; they follow from ordinary unitary quantum mechanics and partial trace.

A.9 Physical realization and resource count

The minimal mixed-environment realization uses one system qubit and three ancillary qubits:

$$N_{\mathrm{qubits}} = 4. \tag{143}$$

The three interactions require three Pauli rotations,

$$R_{XX}(\theta_x), \qquad R_{YY}(\theta_y), \qquad R_{ZZ}(\theta_z), \tag{144}$$

each of which can be decomposed into two CNOT gates and single-qubit Clifford rotations plus one R_z rotation.

Thus the CPUC channel admits a standard circuit implementation using

$$\boxed{\text{CNOT gates} + \text{single-qubit Clifford gates} + R_z \text{ rotations} + \text{ancilla discard.}} \tag{145}$$

If the maximally mixed ancillas are purified, three additional reference qubits are required. The fully unitary purification then acts on

$$1 + 3 + 3 = 7 \tag{146}$$

qubits, corresponding to a $2^7 = 128$-dimensional Hilbert space. The mixed-state four-qubit implementation and the seven-qubit purified implementation describe the same reduced CPUC channel.

A.10 First-principles status of the construction

The circuit construction establishes the following implication:

$$\boxed{\begin{array}{c} \text{Hermitian system–ancilla Hamiltonians} \\ \Downarrow \\ \text{finite-dimensional unitary evolution} \\ \Downarrow \\ \text{system–environment correlations} \\ \Downarrow \quad \mathrm{Tr}_E \\ \text{elementary cosine channels} \\ \Downarrow \\ \lambda_x = bc, \quad \lambda_y = ac, \quad \lambda_z = ab \\ \Downarrow \\ \text{CPUC channel.} \end{array}} \tag{147}$$

Therefore the CPUC map need not be introduced as an independent algebraic ansatz. Its cosine-product Bloch eigenvalues arise from ordinary finite-dimensional unitary quantum dynamics after the environmental degrees of freedom are discarded.

It is important, however, to distinguish this unitary dilation from a time-homogeneous Markovian Lindblad semigroup on the reduced system. The enlarged system obeys the Hamiltonian GKSL equation

$$\frac{d\rho_{SE}}{dt} = -i[H, \rho_{SE}], \tag{148}$$

whereas the reduced CPUC dynamics obtained by tracing out a finite environment is, in general, non-Markovian and need not form a time-homogeneous semigroup. The present construction is therefore a finite-dimensional unitary dilation of the CPUC channel rather than a claim that the reduced cosine dynamics is itself generated by a time-independent Lindblad generator.

This distinction preserves the first-principles interpretation while making explicit the physical origin of the CPUC geometry.

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