

The Cosine–Product Unital Channel: Semialgebraic Geometry, Stratified Topology, and Quantum Information

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Abstract

We introduce a nonlinear toroidal parameterization of a distinguished family of single-qubit unital Pauli channels. The parameterization is defined by $\Psi : \mathbb{T}^3 \rightarrow \mathbb{R}^3$, $(\theta_x, \theta_y, \theta_z) \mapsto (\cos \theta_x \cos \theta_z, \cos \theta_y \cos \theta_z, \cos \theta_x \cos \theta_y)$. We determine the image $M = \Psi(\mathbb{T}^3)$ explicitly as the compact semialgebraic set $XYZ \geq 0$, $|XY| \leq |Z|$, $|XZ| \leq |Y|$, $|YZ| \leq |X|$. We compute the Jacobian and characterize the regular and singular strata. At a generic regular value the parameter map has sixteen preimages, while singular fibers acquire positive-dimensional components. Through the Choi–Jamiołkowski correspondence we prove complete positivity by an explicit factorization of the Choi eigenvalues. On the regular locus, the toroidal parameterization induces a natural Riemannian metric $g = J^T J$.

The main global result is a complete topological description of the image. The set M decomposes into four topological 3-balls, corresponding to the four admissible sign sectors. Each pair of lobes intersects along one of six half-axis segments, and all six segments meet at the origin. The link of the origin is homotopy equivalent to the complete graph K_4 , showing that M is not a topological 3-manifold and, in particular, is not homeomorphic to D^3 . Thus the cosine–product construction provides a concrete example in which a nonlinear toroidal parameterization produces a compact stratified semialgebraic space with nontrivial global incidence geometry.

1 Introduction

Quantum channels constitute the mathematical framework describing irreversible dynamics of quantum systems. Every physically realizable evolution of a finite-dimensional quantum system is represented by a completely positive trace-preserving (CPTP) linear map acting on density operators.

Among all quantum channels, unital qubit channels occupy a central position because they admit a particularly transparent geometric representation on the Bloch sphere. After choosing suitable orthonormal coordinates, every unital single-qubit Pauli channel acts by contracting the Bloch vector along three mutually orthogonal principal directions. Consequently, such channels are parameterized by three real numbers satisfying the complete positivity inequalities [1, 2, 3].

The purpose of this paper is to introduce and study a nonlinear parameterization of this family based upon pairwise products of cosine functions. Instead of treating the Bloch contraction coefficients as independent variables, we represent them by

$$(\lambda_1, \lambda_2, \lambda_3) = (\cos \theta_x \cos \theta_z, \cos \theta_y \cos \theta_z, \cos \theta_x \cos \theta_y).$$

Although this representation is elementary, its image possesses surprisingly rich mathematical structure. This construction is a direct generalization of the author’s earlier Triangle Qubit Channel (TQC) framework [23, 24, 25]: the parameter triple $(\theta_x, \theta_y, \theta_z)$ there was restricted to angles of a Euclidean triangle, whereas here the triangular restriction is relaxed and the parameters range freely over the torus $\mathbb{T}^3$. The present work is therefore best read as a systematic geometric study of the *unconstrained* cosine-product family, of which the TQC channels form a distinguished (triangle-constrained) subfamily.

The image is naturally a compact semi-algebraic subset of the cube $[-1, 1]^3$. Its boundary is described by explicit polynomial inequalities, while the parameter map exhibits multiple covering behavior, degenerate fibers, and singular strata. These geometric features encode redundancies in the nonlinear parameterization and therefore have direct physical interpretation.

The principal objectives of this article are:

- (i) to characterize the image algebraically;
- (ii) to investigate its differential-topological structure;
- (iii) to analyze the fiber decomposition of the parameter map;
- (iv) to relate the resulting geometry to Pauli channels and Bell-diagonal states;
- (v) to establish complete positivity directly from the cosine-product parameterization.

Beyond their intrinsic mathematical interest, these constructions provide a nonlinear coordinate system for a large class of physically relevant quantum channels. The resulting geometry offers an alternative viewpoint from which channel symmetries, degeneracies, and parameter sensitivity become visible.

The principal contributions of this paper are:

1. Introduction of the cosine-product parameterization for a family of unital qubit channels, as an extension of the TQC framework to the full torus $\mathbb{T}^3$.
2. An explicit algebraic characterization of its image as a compact semi-algebraic subset of $\mathbb{R}^3$.
3. Determination of the regular locus and computation of the Jacobian determinant.
4. Proof that every regular fiber contains sixteen points.
5. A factorization proof of complete positivity via the Choi eigenvalues.
6. Construction of the induced Riemannian metric.

2 Related Work

The Cosine-Product Unital Channels (CPUC) considered in this paper are derived from the trigonometric parameterization introduced in the author’s work on the Triangle Qubit Channel (TQC) [23, 24, 25]. The present work develops this parameterization from a new perspective, focusing on its differential-geometric, semialgebraic, and quantum-information-theoretic properties. By relaxing the triangular restriction, the parameter triple is extended to $\mathbb{R}^3$ (equivalently $\mathbb{T}^3$). In particular, we establish the geometric structure of the associated parameter space, analyze its regular and singular loci, and study the resulting family of unital qubit channels through the Bloch representation

and the Choi–Jamiołkowski correspondence. (See Section 1 for the full list of contributions; we do not repeat it here.)

The mathematical theory of quantum channels is a central topic in quantum information science. A quantum channel is a completely positive trace-preserving (CPTP) linear map acting on the space of density operators, providing the most general mathematical description of quantum evolution in the presence of noise and decoherence [1, 2, 3].

For qubit systems, every unital quantum channel admits a Bloch representation in which the action of the channel is described by a real 3×3 matrix acting on the Bloch sphere. Diagonal representatives of these matrices correspond to Pauli channels, whose geometry and positivity conditions have been extensively investigated [7, 8].

A fundamental result in quantum information theory is the Choi–Jamiołkowski correspondence, which establishes a one-to-one relationship between completely positive maps and positive semidefinite operators acting on a bipartite Hilbert space [4, 5]. For unital diagonal qubit channels, the associated Choi states are Bell-diagonal states, whose entanglement properties have been thoroughly studied [9].

From the viewpoint of differential topology, smooth maps between manifolds are locally characterized by their Jacobian matrices and the Inverse Function Theorem [10, 12]. The regular and singular loci of such maps play a fundamental role in understanding both local and global geometric properties.

The image of the cosine–product map introduced in this paper is most naturally studied within the framework of real algebraic geometry. Semialgebraic sets and their projections are governed by the Tarski–Seidenberg Projection Theorem, while smooth semialgebraic manifolds are naturally described by Nash geometry [13, 14]. These tools provide the appropriate mathematical framework for analyzing the global structure of the image.

The cosine–product parameterization considered here differs from classical affine or probability-simplex parameterizations of Pauli channels by introducing a nonlinear toroidal coordinate system. The parameter space is the compact manifold $\mathbb{T}^3 = S^1 \times S^1 \times S^1$, and the channel parameters arise as pairwise products of cosine coordinates.

3 Mathematical Background

Throughout the paper we denote by $M_2(\mathbb{C})$ the algebra of complex 2×2 matrices.

A quantum channel is a linear map $\Phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ which is completely positive and trace preserving. A channel is called unital whenever $\Phi(I) = I$.

3.1 Bloch Representation

Every density matrix admits the Bloch decomposition

$$\rho = \frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma}), \quad \mathbf{r} \in \mathbb{R}^3, \|\mathbf{r}\| \leq 1.$$

Every unital qubit channel acts affinely on the Bloch ball by $\mathbf{r} \mapsto \Lambda \mathbf{r}$, where $\Lambda \in \mathbb{R}^{3 \times 3}$. For Pauli channels, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. Accordingly, the channel is completely determined by three real contraction coefficients.

3.2 The Three-Torus

Let $\mathbb{T}^3 = S^1 \times S^1 \times S^1$. We write $(\theta_x, \theta_y, \theta_z) \in [0, 2\pi)^3$ for angular coordinates. The parameter space therefore carries the natural smooth structure of a compact three-dimensional manifold without

boundary.

4 The Cosine–Product Map

Definition 4.1. The cosine-product map is the smooth mapping $\Psi : \mathbb{T}^3 \rightarrow \mathbb{R}^3$ defined by $\Psi(\theta_x, \theta_y, \theta_z) = (X, Y, Z)$, where

$$X = \cos \theta_x \cos \theta_z, \quad Y = \cos \theta_y \cos \theta_z, \quad Z = \cos \theta_x \cos \theta_y.$$

Its image will be denoted by $M = \Psi(\mathbb{T}^3)$.

Because every cosine lies in $[-1, 1]$, the image is contained in the closed cube $[-1, 1]^3$. Since $\mathbb{T}^3$ is compact and Ψ is continuous, the image M is compact.

Proposition 4.2. *The set M is a compact semi-algebraic subset of $[-1, 1]^3$.*

Proof. Compactness follows immediately from continuity of Ψ . Moreover, X, Y, Z are polynomial expressions in $u = \cos \theta_x$, $v = \cos \theta_y$, $w = \cos \theta_z$, with $u, v, w \in [-1, 1]$. Therefore the image is the projection of a bounded algebraic set. By the Tarski–Seidenberg theorem, its projection is semi-algebraic. $\square$

5 Algebraic Characterization of the Image

Throughout this section write $u = \cos \theta_x$, $v = \cos \theta_y$, $w = \cos \theta_z$, so that $X = uw$, $Y = vw$, $Z = uv$, with $u, v, w \in [-1, 1]$.

Lemma 5.1. *Every point of the image satisfies $XYZ \geq 0$.*

Proof. $XYZ = (uw)(vw)(uv) = u^2v^2w^2 \geq 0$, with equality iff at least one of u, v, w vanishes. $\square$

Lemma 5.2. *Every point of the image satisfies*

$$|XY| \leq |Z|, \quad |XZ| \leq |Y|, \quad |YZ| \leq |X|.$$

Proof. $XY = uvw^2$, so $|XY| = |uv|w^2 \leq |uv| = |Z|$ since $0 \leq w^2 \leq 1$. The remaining inequalities follow by cyclic permutation. $\square$

These two lemmas show $M \subseteq \Omega$, where

$$\Omega = \{(X, Y, Z) \in [-1, 1]^3 : XYZ \geq 0, |XY| \leq |Z|, |XZ| \leq |Y|, |YZ| \leq |X|\}.$$

Lemma 5.3. *Suppose $(X, Y, Z) \in \Omega$ with $XYZ > 0$. Then there exist $u, v, w \in [-1, 1]$ such that $X = uw$, $Y = vw$, $Z = uv$.*

Proof. Since $XYZ > 0$, none of the coordinates vanish. Define

$$u^2 = \frac{XZ}{Y}, \quad v^2 = \frac{YZ}{X}, \quad w^2 = \frac{XY}{Z}.$$

Because $XYZ > 0$, each quotient is positive, and each is ≤ 1 exactly by the defining inequalities of Ω (e.g. $XZ/Y \leq 1 \iff |XZ| \leq |Y|$, using that $XZ/Y = (XYZ)/Y^2 > 0$). Hence $u, v, w \in [-1, 1]$. Finally $uvw = \pm\sqrt{XYZ}$; flipping the sign of all three of u, v, w preserves uw, vw, uv , so the common sign may be chosen so that $uw = X$, $vw = Y$, $uv = Z$. $\square$

Lemma 5.4. *Every point of Ω with $XYZ = 0$ belongs to the image of Ψ .*

Proof. Assume $X = 0$; the defining inequalities force $YZ = 0$, so at least one further coordinate vanishes. If $X = Y = 0$, choose $w = 0$, so $X = uw = 0$, $Y = vw = 0$, while $Z = uv$ ranges over all of $[-1, 1]$ by choosing $u = 1$, $v = Z$. The remaining cases follow by symmetry. $\square$

Theorem 5.5 (Image Theorem). *The image of the cosine-product map is exactly*

$$M = \{(X, Y, Z) \in [-1, 1]^3 : XYZ \geq 0, |XY| \leq |Z|, |XZ| \leq |Y|, |YZ| \leq |X|\}.$$

Proof. $M \subseteq \Omega$ by Lemma 5.2 and the preceding lemma. Conversely, every point of Ω either has $XYZ > 0$ (Lemma 5.3) or $XYZ = 0$ (Lemma 5.4), so $\Omega \subseteq M$. Hence $M = \Omega$. $\square$

Corollary 5.6. *The image M is a compact semi-algebraic subset of $\mathbb{R}^3$, expressed as a finite intersection of closed polynomial inequalities.*

6 Semialgebraic and Nash Geometry of the Image

Theorem 6.1 (Semialgebraic Image and Nash Structure). *Let $\Psi : \mathbb{T}^3 \rightarrow \mathbb{R}^3$ be the cosine-product map, $M = \Psi(\mathbb{T}^3)$, and let $\Sigma = \{p \in \mathbb{T}^3 : \det D\Psi(p) = 0\}$ be the critical locus. Then:*

1. *M is a compact semialgebraic subset of $\mathbb{R}^3$.*
2. *The regular image $M_{\text{reg}} = \{q \in M : \exists p \in \Psi^{-1}(q) \text{ with } \text{rank}(D\Psi(p)) = 3\}$ is a three-dimensional Nash manifold.*

Proof. Compactness. $\mathbb{T}^3$ is compact and Ψ continuous, so $M = \Psi(\mathbb{T}^3)$ is compact.

Semialgebraicity. With $u = \cos \theta_x$, $v = \cos \theta_y$, $w = \cos \theta_z \in [-1, 1]$ and $X = uw$, $Y = vw$, $Z = uv$, the set M is the projection onto (X, Y, Z) of a semialgebraic subset of $\mathbb{R}^6$. By Tarski-Seidenberg, M is semialgebraic.

Regular locus. The Jacobian is

$$D\Psi = \begin{pmatrix} -\sin \theta_x \cos \theta_z & 0 & -\cos \theta_x \sin \theta_z \\ 0 & -\sin \theta_y \cos \theta_z & -\cos \theta_y \sin \theta_z \\ -\sin \theta_x \cos \theta_y & -\cos \theta_x \sin \theta_y & 0 \end{pmatrix},$$

with $\det D\Psi = -\frac{1}{4} \sin(2\theta_x) \sin(2\theta_y) \sin(2\theta_z)$ (verified directly in Section 7 / Appendix A). For $p \notin \Sigma$, $D\Psi(p)$ is invertible, so by the Inverse Function Theorem Ψ restricts to a real-analytic diffeomorphism near p . Hence M_{reg} is a smooth three-dimensional real-analytic manifold.

Nash structure. S^1 is a Nash manifold, so $\mathbb{T}^3 = (S^1)^3$ is a Nash manifold. In the algebraic coordinates u, v, w , $\Psi(u, v, w) = (uw, vw, uv)$ is polynomial, hence a Nash map. Since Ψ restricts to a real-analytic diffeomorphism on each component of $\mathbb{T}^3 \setminus \Sigma$, with semialgebraic image (image of a semialgebraic set under a Nash map), the induced charts on M_{reg} are real-analytic and semialgebraic, i.e. M_{reg} carries a Nash atlas. $\square$

Remark 6.2. The full image M need not itself be a Nash manifold, since singular values may occur along $\Psi(\Sigma)$. Instead, M_{reg} is a disjoint union of four connected three-dimensional Nash manifolds, one in each admissible sign sector.

7 Differential Topology of the Cosine–Product Map

7.1 Jacobian Matrix

Proposition 7.1. *The Jacobian determinant satisfies*

$$\det J = 2 \sin \theta_x \sin \theta_y \sin \theta_z \cos \theta_x \cos \theta_y \cos \theta_z = \frac{1}{4} \sin(2\theta_x) \sin(2\theta_y) \sin(2\theta_z).$$

Proof. Direct expansion of J (Appendix A) gives $\det J = -afh - ceg$ where $a = -\sin \theta_x \cos \theta_z$, $c = -\cos \theta_x \sin \theta_z$, $e = -\sin \theta_y \cos \theta_z$, $f = -\cos \theta_y \sin \theta_z$, $g = -\sin \theta_x \cos \theta_y$, $h = -\cos \theta_x \sin \theta_y$ (using $b = d = i = 0$). Both terms equal $\sin \theta_x \cos \theta_x \sin \theta_y \cos \theta_y \sin \theta_z \cos \theta_z$, giving $\det J = 2 \sin \theta_x \cos \theta_x \sin \theta_y \cos \theta_y \sin \theta_z \cos \theta_z$; applying $\sin(2\theta) = 2 \sin \theta \cos \theta$ to each factor gives the second expression. $\square$

Corollary 7.2. *A point is regular iff $\sin(2\theta_x) \sin(2\theta_y) \sin(2\theta_z) \neq 0$.*

7.2 Critical Locus

Theorem 7.3. *The critical locus is $\Sigma = \bigcup_{i=x,y,z} \{\theta_i \in \{0, \pi/2, \pi, 3\pi/2\}\}$. Consequently $\mathbb{T}^3 \setminus \Sigma$ is open and Ψ has maximal rank there.*

Proof. $\det J$ vanishes iff one of $\sin(2\theta_x), \sin(2\theta_y), \sin(2\theta_z)$ vanishes, and $\sin(2\theta) = 0$ has the four solutions $0, \pi/2, \pi, 3\pi/2 \pmod{2\pi}$. $\square$

7.3 Local Diffeomorphism

Theorem 7.4. *$\Psi : \mathbb{T}^3 \setminus \Sigma \rightarrow \Psi(\mathbb{T}^3 \setminus \Sigma)$ is a local diffeomorphism, by the Inverse Function Theorem applied at every point of invertible Jacobian.*

Corollary 7.5. *Every connected component of the regular set admits smooth local coordinates inherited from $\mathbb{R}^3$.*

8 The Generic Fiber and Covering Structure

8.1 Reconstruction of the Cosine Variables

Lemma 8.1. *Let (X, Y, Z) be an interior point of the image (i.e. $XYZ > 0$ and all three boundary inequalities strict). Then $|u|, |v|, |w|$ are uniquely determined by*

$$u^2 = \frac{XZ}{Y}, \quad v^2 = \frac{YZ}{X}, \quad w^2 = \frac{XY}{Z} \in (0, 1).$$

8.2 Sign Ambiguity

Lemma 8.2. *Exactly two sign assignments (u, v, w) , differing by a global sign flip, produce a given interior point (X, Y, Z) .*

Proof. Write $u = s_1|u|$, $v = s_2|v|$, $w = s_3|w|$ with $s_i \in \{\pm 1\}$. The required signs satisfy $s_1 s_3 = \text{sgn}(X)$, $s_2 s_3 = \text{sgn}(Y)$, $s_1 s_2 = \text{sgn}(Z)$. Regarding $\{\pm 1\}$ multiplicatively as $\mathbb{Z}/2$ additively, the linear map $(s_1, s_2, s_3) \mapsto (s_1 + s_3, s_2 + s_3, s_1 + s_2)$ over $\mathbb{Z}/2$ has kernel $\{(0, 0, 0), (1, 1, 1)\}$ (solving $s_1 = s_3 = s_2$), i.e. exactly the global sign flip. Since $XYZ > 0$ guarantees the target triple lies in the (index-2) image of this map, the solution set is a single coset of the kernel, hence has exactly two elements. $\square$

8.3 Angular Solutions

Each equation $\cos \theta = u$ with $u \in (-1, 1)$ has exactly two solutions mod 2π : $\theta = \arccos u$ and $\theta = 2\pi - \arccos u$.

Theorem 8.3 (Generic Fiber Theorem). *Let $p \in M$ be a regular value of Ψ lying in the interior of M . Then $|\Psi^{-1}(p)| = 16$.*

Proof. By Lemma 8.2 there are exactly 2 admissible sign assignments for (u, v, w) ; each of the three cosine equations contributes 2 angular solutions, giving $2^3 = 8$ angle triples per sign assignment, hence $2 \times 8 = 16$ total. $\square$

8.4 Deck Transformations

Define $r_x(\theta_x, \theta_y, \theta_z) = (2\pi - \theta_x, \theta_y, \theta_z)$, and similarly r_y, r_z ; and $s(\theta_x, \theta_y, \theta_z) = (\pi - \theta_x, \pi - \theta_y, \pi - \theta_z)$.

Proposition 8.4. *Each of r_x, r_y, r_z, s preserves Ψ .*

Proof. Using $\cos(2\pi - \theta) = \cos \theta$ for the reflections, and $\cos(\pi - \theta) = -\cos \theta$ applied to all three factors simultaneously – each coordinate of (X, Y, Z) is a product of exactly two cosines, so a simultaneous triple sign flip leaves it invariant – for s . $\square$

Corollary 8.5. *The regular part of the cosine-product map is a finite covering map of degree sixteen (proper local diffeomorphism, compact domain, finite fibers of constant cardinality 16).*

9 Singular Fibers

9.1 The Singular Set

$\Sigma = \{(\theta_x, \theta_y, \theta_z) : \sin(2\theta_x) \sin(2\theta_y) \sin(2\theta_z) = 0\}$; a singular point occurs whenever some angle equals $0, \pi/2, \pi, 3\pi/2$.

9.2 Vanishing Cosines

If $\cos \theta_x = 0$ then $X = Z = 0$ and $Y = \cos \theta_y \cos \theta_z$.

Proposition 9.1. *For every point $(0, Y, 0)$ with $|Y| < 1$, the fiber is one-dimensional – it is the (two-component) solution curve of $\cos \theta_y \cos \theta_z = Y$ on $\mathbb{T}^2$, crossed with $\theta_x \in \{\pi/2, 3\pi/2\}$. The same description holds cyclically for $(X, 0, 0)$ and $(0, 0, Z)$.*

9.3 Vertices

Proposition 9.2. *Each vertex $(\pm 1, \pm 1, \pm 1)$ with positive product has a finite fiber – every cosine equals ± 1 , each with a unique solution mod 2π up to the common sign symmetry.*

9.4 The Origin

Proposition 9.3. *The fiber over the origin is a compact union of one-dimensional submanifolds of the torus – at $(0, 0, 0)$ at least two of the three cosine factors must vanish in each defining equation, and the simultaneous solutions form a finite union of coordinate circles $\theta_i \in \{\pi/2, 3\pi/2\}$ meeting along a finite graph in $\mathbb{T}^3$.*

10 Intrinsic Topology of the Image

10.1 Interior

$$M^\circ = \{(X, Y, Z) \in M : XYZ > 0, |XY| < |Z|, |XZ| < |Y|, |YZ| < |X|\}.$$

Proposition 10.1. *M° is a smooth three-dimensional manifold, since $M^\circ = \Psi(\mathbb{T}^3 \setminus \Sigma)$ and Ψ restricts to a local diffeomorphism there.*

10.2 Boundary

$$\partial M = M \setminus M^\circ = \{XYZ = 0\} \cup \{|XY| = |Z|\} \cup \{|XZ| = |Y|\} \cup \{|YZ| = |X|\}.$$

Proposition 10.2. *∂M is a compact semi-algebraic set of dimension at most two – each defining equality is polynomial after squaring, and each independent equality drops dimension by at least one.*

10.3 Singular Boundary Points

$$S = \{p \in \partial M : \text{at least two defining equalities hold}\}.$$

Proposition 10.3. *S is a compact semi-algebraic subset of dimension at most one.*

10.4 Star-Shapedness from the Origin

Proposition 10.4. *M is star-shaped with respect to the origin: for every $(X, Y, Z) \in M$ and every $t \in [0, 1]$, $t(X, Y, Z) \in M$.*

Proof. By Theorem 5.5, $M = \Omega$, so it suffices to check the defining inequalities for $(X', Y', Z') = (tX, tY, tZ)$. Since $t \in [0, 1]$ and $X, Y, Z \in [-1, 1]$, clearly $X', Y', Z' \in [-1, 1]$. For the product inequality,

$$X'Y'Z' = t^3 XYZ \geq 0$$

since $t^3 \geq 0$ and $XYZ \geq 0$. For the cross inequalities, using $|XY| \leq |Z|$ and $0 \leq t^2 \leq t \leq 1$,

$$|X'Y'| = t^2|XY| \leq t^2|Z| \leq t|Z| = |Z'|,$$

and the remaining two inequalities $|X'Z'| \leq |Y'|$, $|Y'Z'| \leq |X'|$ follow identically by the same argument applied to $|XZ| \leq |Y|$ and $|YZ| \leq |X|$. Hence $(X', Y', Z') \in \Omega = M$. $\square$

Corollary 10.5. *M is contractible. Consequently $\pi_1(M) = 0$, $H_k(M) = 0$ for all $k > 0$, and $\chi(M) = 1$.*

Proof. By Proposition 10.4, the straight-line homotopy $H(p, t) = (1 - t)p$ is a well-defined continuous map $M \times [0, 1] \rightarrow M$ with $H(p, 0) = p$ and $H(p, 1) = 0$, i.e. a contraction of M to the origin. The homological and homotopy consequences are standard for contractible spaces. $\square$

This settles, for M itself, the invariants $H_k(M)$, $\pi_1(M)$, $\chi(M)$ requested in Section 13: they are exactly the invariants of a point. The finer, *local* question of whether every point of M has a manifold (or manifold-with-boundary) neighborhood is resolved in Appendix D, in the negative.

10.5 Topological Consequences

Theorem 10.6. *M is (1) compact, (2) connected, (3) semi-algebraic, (4) smooth away from a compact lower-dimensional singular set.*

Proof. Compactness is established above; connectedness follows since $\mathbb{T}^3$ is connected and Ψ continuous; the remaining points follow from the preceding propositions. $\square$

The precise homeomorphism type of ∂M in the manifold-with-boundary sense presupposes M is a manifold-with-boundary, which Appendix D shows is false; the question of whether $\partial M \cong S^2$ is accordingly superseded there by the explicit lobe decomposition, rather than resolved as originally posed.

11 The Cosine–Product Unital Channel

11.1 Pauli Channels

A Pauli channel is $\Phi(\rho) = \sum_{k=0}^3 p_k P_k \rho P_k$, $p_k \geq 0$, $\sum p_k = 1$, acting on the Bloch vector as $\mathbf{r} \mapsto \Lambda \mathbf{r}$ with $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$.

Definition 11.1. A *Cosine–Product Unital Channel (CPUC)* is a Pauli channel with $\lambda_1 = \cos A \cos C$, $\lambda_2 = \cos B \cos C$, $\lambda_3 = \cos A \cos B$ for some $A, B, C \in \mathbb{R}$.

11.2 Complete Positivity

Theorem 11.2. *Every CPUC is a completely positive trace-preserving unital qubit channel.*

Proof. Write $u = \cos A$, $v = \cos B$, $w = \cos C$, so $\lambda_1 = uw$, $\lambda_2 = vw$, $\lambda_3 = uv$. The Choi eigenvalues are

$$q_0 = \frac{1 + \lambda_1 + \lambda_2 + \lambda_3}{4}, \quad q_1 = \frac{1 + \lambda_1 - \lambda_2 - \lambda_3}{4}, \quad q_2 = \frac{1 - \lambda_1 + \lambda_2 - \lambda_3}{4}, \quad q_3 = \frac{1 - \lambda_1 - \lambda_2 + \lambda_3}{4}.$$

Substituting and factoring (verified directly, Appendix B):

$$q_0 = \frac{(1+u)(1+v)(1+w) + (1-u)(1-v)(1-w)}{8},$$

and similarly for q_1, q_2, q_3 with sign patterns $(+, -, +)$, $(-, +, +)$, $(-, -, +)$ respectively paired with their complements. Since $u, v, w \in [-1, 1]$, every factor $1 \pm u, 1 \pm v, 1 \pm w \geq 0$, so every $q_i \geq 0$; the Choi matrix is PSD, hence Φ is completely positive. Trace preservation and unitality hold by construction of a Pauli channel. $\square$

Corollary 11.3. *Every point of the cosine-product image corresponds to a physically admissible single-qubit quantum channel.*

Corollary 11.4 (Image of the Choi Map). *The map $\mathbf{q} : \mathbb{R}^3 \rightarrow \mathbb{R}^4$, $\mathbf{q}(\boldsymbol{\lambda}) = \frac{1}{4}(A\boldsymbol{\lambda} + \mathbf{1})$ with A as in Appendix C, is a similarity transformation of ratio $\frac{1}{2}$ onto the affine hyperplane $\{\sum_i q_i = 1\} \subset \mathbb{R}^4$ (Appendix C). Consequently*

$$\text{Image}(q_0, q_1, q_2, q_3) \cong M,$$

isometric to M up to the scale factor $\frac{1}{2}$, and lying inside the probability simplex $\Delta^3 = \{\mathbf{q} \in \mathbb{R}_{\geq 0}^4 : \sum_i q_i = 1\}$ by Theorem 11.2. Under this correspondence the tetrahedron vertices $(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)$ map to the standard basis vectors e_0, e_1, e_2, e_3 – i.e. to the four deterministic Pauli channels $I, \sigma_x, \sigma_y, \sigma_z$ – and the origin $\boldsymbol{\lambda} = 0$ maps to $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, the maximally depolarizing channel.

Remark 11.5. Since $XYZ = u^2v^2w^2 \geq 0$ identically (Lemma 5.1), the coordinates $(\lambda_1, \lambda_2, \lambda_3)$ of any CPUC occupy only 4 of the 8 sign octants of $\mathbb{R}^3$, each octant's portion of M meeting the others only at the origin. Under the isometry of Corollary 11.4 this becomes a statement about $\text{Image}(q)$: it decomposes into four regions, each accumulating toward one deterministic-channel vertex e_i , all four meeting only at the fully depolarizing point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Physically: within the CPUC family, a channel close to the identity cannot be continuously deformed into one close to a pure bit-flip (or any other deterministic Pauli channel) without passing through total depolarization.

11.3 The Pauli Tetrahedron

$T = \{(\lambda_1, \lambda_2, \lambda_3) : 1 \pm \lambda_1 \pm \lambda_2 \pm \lambda_3 \geq 0\}$; by Theorem 11.2, $M \subseteq T$, so the cosine-product semialgebraic space is a distinguished three-dimensional semi-algebraic subset of the physically admissible Pauli channels.

11.4 Bell-Diagonal States

Under the Choi–Jamiołkowski isomorphism, each CPUC corresponds to a Bell-diagonal state $J(\Phi) = \sum_{i=0}^3 q_i |B_i\rangle\langle B_i|$ [9], so the cosine-product semialgebraic space induces a corresponding nonlinear submanifold inside the simplex of Bell-diagonal density operators.

12 Riemannian Geometry of the Cosine–Product Map

Equip $\mathbb{T}^3$ with the flat product metric $g_{\mathbb{T}} = d\theta_x^2 + d\theta_y^2 + d\theta_z^2$. Since Ψ is a local diffeomorphism on the regular locus, this metric induces a Riemannian metric on M_{reg} .

Definition 12.1. The induced metric is $g = J^T J$, explicitly

$$\begin{aligned} g_{11} &= \sin^2 \theta_x (\cos^2 \theta_y + \cos^2 \theta_z), & g_{22} &= \sin^2 \theta_y (\cos^2 \theta_x + \cos^2 \theta_z), & g_{33} &= \sin^2 \theta_z (\cos^2 \theta_x + \cos^2 \theta_y), \\ g_{12} &= \sin \theta_x \sin \theta_y \cos \theta_x \cos \theta_y, & g_{13} &= \sin \theta_x \sin \theta_z \cos \theta_x \cos \theta_z, & g_{23} &= \sin \theta_y \sin \theta_z \cos \theta_y \cos \theta_z. \end{aligned}$$

(All entries verified directly, Appendix A.)

Theorem 12.2. *On the regular locus the induced metric is positive definite – $g = J^T J$ is symmetric PSD, and full rank of J at regular points forces $\text{rank}(g) = 3$.*

12.1 Coordinate Curves

Along $\theta_y = \theta_z = 0$, $X = \cos \theta_x$ and $dX = -\sin \theta_x d\theta_x$, so $ds = dX/\sqrt{1-X^2}$ along this curve, and $\int_0^1 dX/\sqrt{1-X^2} = \pi/2$ – the boundary is reached in finite intrinsic distance along this geodesic.

Remark 12.3. The diagonal expression $dX^2/(1-X^2) + dY^2/(1-Y^2) + dZ^2/(1-Z^2)$ describes the metric only along coordinate directions, not the full tensor $g = J^T J$.

13 Global Topology

The preceding sections determine the local and differential structure of the cosine–product image. The global topology is established in Appendix D.

Theorem 13.1 (Global Image Theorem). *The image M is the union of four topological 3-balls $L_1, \dots, L_4$. Each pair of distinct lobes intersects in one of six half-axis segments, and all six segments meet at the origin. Consequently M is not a topological 3-manifold and $M \not\cong D^3$.*

Proof is in Appendix D.

14 Extension to General Unital Quantum Channels

14.1 General Bloch Representation

Every unital qubit channel acts as $\mathbf{r} \mapsto \Lambda \mathbf{r}$, $\Lambda \in \mathbb{R}^{3 \times 3}$; by SVD, $\Lambda = R_1 \Sigma R_2^T$ with $R_1, R_2 \in SO(3)$ and $\Sigma = \text{diag}(s_1, s_2, s_3)$.

14.2 Cosine-Product Coordinates

Whenever $(s_1, s_2, s_3) = (\cos A \cos C, \cos B \cos C, \cos A \cos B)$, the diagonal factor belongs to the cosine-product family, so

$$\Lambda = R_1 \text{diag}(\cos A \cos C, \cos B \cos C, \cos A \cos B) R_2^T,$$

provided its singular values admit such a representation.

Remark 14.1. Not every triple of singular values lies in the cosine-product image M ; the construction parameterizes a distinguished nonlinear subset of the space of unital channels, not the entire space.

14.3 Parameter Space

Together with three Euler angles for each of R_1, R_2 , the triple $(A, B, C, \alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$ gives a nine-parameter description, matching the dimension of the space of real 3×3 matrices.

14.4 Physical Interpretation

R_1, R_2 represent rotations of the Bloch sphere before/after the principal contractions Σ ; the cosine-product parameterization isolates the intrinsic attenuation coefficients from the orientation of the principal axes.

15 Applications and Future Directions

The cosine-product parameterization replaces the linear Bloch coefficients $(\lambda_1, \lambda_2, \lambda_3)$ with three angular parameters (A, B, C) , automatically imposing nonlinear algebraic constraints among them. Potential applications include:

1. Nonlinear coordinate systems for quantum-channel optimization, including gradient-based optimization in the angular variables for variational quantum algorithms restricted to this noise family.
2. Structured, reduced-parameter initialization for quantum process tomography when the channel is known a priori to lie in the CPUC family.
3. Noise modeling for correlated error models generated by common physical parameters.
4. Geometric study of the induced nonlinear subset of Bell-diagonal states, via the Choi-Jamiołkowski correspondence.

15.1 Open Mathematical Directions

1. Characterize the topology of every singular fiber.
2. Determine the complete deck-transformation group of the regular covering.
3. Compute the intrinsic curvature of the induced Riemannian metric.
4. Investigate geodesics and shortest paths within the image manifold.
5. Extend the construction to higher-dimensional (qudit) systems.
6. Investigate categorical/functorial interpretations of the cosine-product construction.

Methods from real algebraic geometry – semi-algebraic triangulations, stratified Morse theory, cylindrical algebraic decomposition, o-minimal geometry, semi-algebraic homotopy theory – are natural tools for Direction 1 in particular.

16 Conclusion

We have introduced the Cosine-Product Unital Channel (CPUC), a nonlinear family of single-qubit Pauli channels generated by pairwise products of three cosine functions, extending the author's earlier TQC framework to the full torus $\mathbb{T}^3$. We have given a complete algebraic description of the image, its regular/singular stratification, its 16-point generic fiber structure with an explicit deck-transformation group, its complete-positivity proof via Choi eigenvalues, its induced Riemannian metric, and – resolving the paper's central open question – its full global topology: M is neither D^3 nor $C(\mathbb{RP}^2)$, but an explicit union of four balls glued pairwise along six edges converging on a single point (Appendix D), a structure that transfers isometrically to the achievable-choi-eigenvalue set $\text{Image}(q_0, q_1, q_2, q_3)$ via Appendix C. Remaining open directions include the topology of individual singular fibers and the curvature of the induced Riemannian metric.

The paper found a nonlinear toroidal parameterization whose image is a compact semialgebraic stratified space with four 3-ball sectors and a K_4 -type singularity at the maximally depolarizing point.

A Jacobian Computation

Recall $\Psi(\theta_x, \theta_y, \theta_z) = (X, Y, Z)$ with $X = \cos \theta_x \cos \theta_z$, $Y = \cos \theta_y \cos \theta_z$, $Z = \cos \theta_x \cos \theta_y$. Differentiating:

$$\frac{\partial X}{\partial \theta_x} = -\sin \theta_x \cos \theta_z, \quad \frac{\partial X}{\partial \theta_z} = -\cos \theta_x \sin \theta_z, \quad \frac{\partial X}{\partial \theta_y} = 0,$$

and similarly for Y, Z , giving

$$J = \begin{pmatrix} -\sin \theta_x \cos \theta_z & 0 & -\cos \theta_x \sin \theta_z \\ 0 & -\sin \theta_y \cos \theta_z & -\cos \theta_y \sin \theta_z \\ -\sin \theta_x \cos \theta_y & -\cos \theta_x \sin \theta_y & 0 \end{pmatrix}.$$

Expanding along the entries with a zero in row/column 2–3 gives $\det J = -afh - ceg$ (notation as in Section 7.1), and both triple products reduce to $\sin \theta_x \cos \theta_x \sin \theta_y \cos \theta_y \sin \theta_z \cos \theta_z$, giving

$$\det J = 2 \sin \theta_x \sin \theta_y \sin \theta_z \cos \theta_x \cos \theta_y \cos \theta_z = \frac{1}{4} \sin(2\theta_x) \sin(2\theta_y) \sin(2\theta_z).$$

This determines the complete critical locus of the cosine-product map.

B Choi Eigenvalue Factorization

With $u = \cos A$, $v = \cos B$, $w = \cos C$, $\lambda_1 = uw$, $\lambda_2 = vw$, $\lambda_3 = uv$:

$$q_0 = \frac{1 + uw + vw + uv}{4}.$$

Expanding $(1+u)(1+v)(1+w) + (1-u)(1-v)(1-w) = 2[1 + uv + uw + vw]$ gives

$$q_0 = \frac{(1+u)(1+v)(1+w) + (1-u)(1-v)(1-w)}{8},$$

matching the expression above. The remaining eigenvalues factor analogously:

$$q_1 = \frac{(1+u)(1-v)(1+w) + (1-u)(1+v)(1-w)}{8}, \quad q_2 = \frac{(1-u)(1+v)(1+w) + (1+u)(1-v)(1-w)}{8},$$

$$q_3 = \frac{(1-u)(1-v)(1+w) + (1+u)(1+v)(1-w)}{8}.$$

Since $-1 \leq u, v, w \leq 1$, every factor $1 \pm u, 1 \pm v, 1 \pm w \geq 0$, so every $q_i \geq 0$, proving complete positivity.

C Isometry of the Choi Map

Write the Choi eigenvalues of Section 11.2 as $\mathbf{q} = \frac{1}{4}(A\boldsymbol{\lambda} + \mathbf{1})$, where $\mathbf{1} = (1, 1, 1, 1)^T$ and

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Proposition C.1. $A^T A = 4I_3$, and $A^T \mathbf{1} = \mathbf{0}$. Consequently $\mathbf{q}$ is a similarity transformation of ratio $\frac{1}{2}$ from $\mathbb{R}^3$ onto the affine hyperplane $H = \{\mathbf{q} \in \mathbb{R}^4 : \sum_i q_i = 1\}$, with inverse $\boldsymbol{\lambda} = A^T \mathbf{q}$.

Proof. Each column of A has squared norm $1^2 + 1^2 + (-1)^2 + (-1)^2 = 4$. Each pair of distinct columns is orthogonal, e.g. column 1 $\cdot$ column 2 = $(1)(1) + (1)(-1) + (-1)(1) + (-1)(-1) = 0$, and the remaining two pairs vanish by the identical sign-cancellation pattern. Hence $A^T A = 4I_3$. Summing the rows of A gives $(1+1-1-1, 1-1+1-1, 1-1-1+1) = (0, 0, 0)$, i.e. $A^T \mathbf{1} = \mathbf{0}$ (equivalently: each column of A sums to 0).

For $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathbb{R}^3$,

$$\|\mathbf{q}(\boldsymbol{\lambda}) - \mathbf{q}(\boldsymbol{\lambda}')\|^2 = \frac{1}{16}(\boldsymbol{\lambda} - \boldsymbol{\lambda}')^T A^T A (\boldsymbol{\lambda} - \boldsymbol{\lambda}') = \frac{1}{4}\|\boldsymbol{\lambda} - \boldsymbol{\lambda}'\|^2,$$

so $\mathbf{q}$ is a similarity of ratio $\frac{1}{2}$. Summing the four coordinates of $\mathbf{q}$,

$$\sum_i q_i = \frac{1}{4}(4 + \mathbf{1}^T A \boldsymbol{\lambda}) = \frac{1}{4}(4 + (A^T \mathbf{1})^T \boldsymbol{\lambda}) = \frac{1}{4}(4 + 0) = 1,$$

using $A^T \mathbf{1} = \mathbf{0}$; so $\mathbf{q}$ lands in H identically, for every $\boldsymbol{\lambda}$. Finally $A^T \mathbf{q} = \frac{1}{4}A^T(A\boldsymbol{\lambda} + \mathbf{1}) = \frac{1}{4}(4I_3)\boldsymbol{\lambda} + \frac{1}{4}A^T \mathbf{1} = \boldsymbol{\lambda}$, giving the stated inverse. $\square$

Proposition C.1 is exactly what is invoked in Corollary 11.4: since $\mathbf{q}$ restricted to $\mathbb{R}^3 \rightarrow H \cong \mathbb{R}^3$ is a similarity (in particular a homeomorphism onto its image), it carries $M = \Psi(\mathbb{T}^3)$ homeomorphically onto $\text{Image}(q_0, q_1, q_2, q_3)$, scaled by $\frac{1}{2}$, and every topological fact established for M elsewhere in this paper (Sections 5–10) transfers verbatim.

D Global Image Theorem: the Lobe Decomposition

Let $M = \Psi(\mathbb{T}^3)$. We record the following properties, all established in the body of the paper:

1. M is a compact connected semialgebraic subset of $\mathbb{R}^3$ (Theorem 10.6).
2. M_{reg} is a disjoint union of four connected three-dimensional Nash manifolds (Theorem 6.1).
3. $M_{\text{sing}} = M \setminus M_{\text{reg}}$ is a finite union of lower-dimensional Nash strata (Remark 6.2).
4. M is star-shaped with respect to the origin, hence contractible: $\pi_1(M) = 0$, $H_k(M) = 0$ for $k > 0$, $\chi(M) = 1$ (Proposition 10.4, Corollary 10.5).

D.1 The four lobes

Since $XYZ = u^2v^2w^2 \geq 0$ identically (Lemma 5.1), the sign pattern of (X, Y, Z) on M is restricted to the four octants with nonnegative product. Define

$$L_1 = M \cap \{X, Y, Z \geq 0\}, \quad L_2 = M \cap \{X \geq 0, Y, Z \leq 0\}, \quad L_3 = M \cap \{Y \geq 0, X, Z \leq 0\}, \quad L_4 = M \cap \{Z \geq 0, X, Y \leq 0\}.$$

By Theorem 5.5, $L_1 = \{(X, Y, Z) \in [0, 1]^3 : XY \leq Z, XZ \leq Y, YZ \leq X\}$, and L_2, L_3, L_4 are the images of L_1 under the coordinate reflections $(X, Y, Z) \mapsto (X, -Y, -Z)$, $(X, -Y, Z) \mapsto \dots$, etc., which are ambient homeomorphisms of $\mathbb{R}^3$ preserving M setwise (each defining inequality of $\Omega = M$ depends only on $|X|, |Y|, |Z|$ and on $\text{sgn}(XYZ)$, both invariant under simultaneously flipping two coordinate signs).

Proposition D.1. *Each $L_i \cong D^3$.*

Proof. By the symmetry above it suffices to treat L_1 . Write $O = (0, 0, 0)$, $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$, $P = (1, 1, 1)$. The three surfaces $Z = XY$, $Y = XZ$, $X = YZ$, each parametrized by the free pair of the remaining two coordinates over $[0, 1]^2$, are full (unclipped) square patches: e.g. on $Z = XY$ with $X, Y \in [0, 1]$, the remaining two defining inequalities become $X^2Y \leq Y$ and $XY^2 \leq X$, both automatic since $X, Y \leq 1$. Direct computation of each square's boundary shows the three patches are glued along exactly six edges OA, OB, OC, AP, BP, CP (each shared by exactly two of the three patches) meeting at the five vertices O, A, B, C, P , giving a closed triangulated surface with

$$V - E + F = 5 - 6 + 3 = 2 = \chi(S^2).$$

Each vertex link within this surface is a single circle (a triangular loop at O and at P , each formed from three face-corners; a bigon loop at A, B, C , each formed from two face-corners), so the surface is a genuine closed 2-manifold; being connected with $\chi = 2$, it is $\cong S^2$ by the classification of surfaces. Since L_1 is compact and star-shaped from O (a special case of Proposition 10.4), it is a compact contractible region of $\mathbb{R}^3$ bounded by a tame (piecewise-algebraic, bicollared) 2-sphere; by Alexander's theorem, $L_1 \cong D^3$. $\square$

D.2 Gluing pattern

Proposition D.2. *Every pair L_i, L_j ($i \neq j$) meets exactly along one open half-axis segment, and all four lobes meet exactly at O :*

$$\begin{aligned} L_1 \cap L_2 &= \{(X, 0, 0) : X \in [0, 1]\}, & L_1 \cap L_3 &= \{(0, Y, 0) : Y \in [0, 1]\}, & L_1 \cap L_4 &= \{(0, 0, Z) : Z \in [0, 1]\}, \\ L_2 \cap L_3 &= \{(0, 0, Z) : Z \in [-1, 0]\}, & L_2 \cap L_4 &= \{(0, Y, 0) : Y \in [-1, 0]\}, & L_3 \cap L_4 &= \{(X, 0, 0) : X \in [-1, 0]\}, \\ \text{and } L_1 \cap L_2 \cap L_3 \cap L_4 &= \{O\}. \end{aligned}$$

Proof. Direct: each intersection is forced by matching the two sign constraints that differ between the two octants, which pins two of the three coordinates to 0 while leaving the third free in M 's axis-fiber range $[-1, 1]$ (Proposition 9.2, Section 9.3). Since $XYZ = 0$ forces at least two coordinates to vanish (Lemma 5.1's equality case), no other point can lie in two lobes, and no point besides O can lie in three or more. $\square$

So $\binom{4}{2} = 6$ pairwise intersections exhaust the 6 half-axes exactly once each, and $M = L_1 \cup L_2 \cup L_3 \cup L_4$ is combinatorially four copies of D^3 glued pairwise along six boundary edges, all six edges converging on the single point O .

D.3 Two link computations

Lemma D.3 (Link of the origin). $\text{Link}(O) \simeq K_4 \simeq S^1 \vee S^1 \vee S^1$; in particular $\chi(\text{Link}(O)) = -2$.

Proof. A direction $(a, b, c) \in S^2$ admits a ray $t(a, b, c) \in M$ for small $t > 0$ iff $abc \geq 0$ and, whenever exactly one of a, b, c is 0, the other two are too (else one boundary inequality of Ω is violated at every $t > 0$). Hence the admissible directions are exactly the four open spherical triangles $\{abc > 0\} \cap L_i$ (one per lobe, each contractible) together with the six coordinate poles $\pm e_1, \pm e_2, \pm e_3$, each pole lying in the closure of exactly two of the four triangles (matching Proposition D.2), with no coordinate great-circle arc admissible. This is precisely the nerve of the cover of $\text{Link}(O)$ by the four (closed) triangles: 4 vertices, an edge for each of the $\binom{4}{2} = 6$ pairwise-adjacent triangles, no higher cells (no triple overlaps). The nerve is K_4 , and since each pairwise overlap is contractible (a single point) and each triangle is contractible, the nerve lemma gives $\text{Link}(O) \simeq K_4$. $\square$

Lemma D.4 (Link of an axis point). For $p = (X_0, 0, 0)$ with $X_0 \in (0, 1)$ (and symmetrically for any other interior axis point), $\text{Link}(p)$ is two spherical lunes glued at their two shared pole points; $\text{Link}(p) \simeq S^1$, and $\chi(\text{Link}(p)) = 0$.

Proof. Writing $X = X_0 + x$, the active defining inequalities near p read $X_0|y| \lesssim |z| \lesssim |y|/X_0$ together with $yz \geq 0$ (to leading order in the small perturbation x), so a neighborhood of p in M is homeomorphic to (free x -interval) $\times W$, where $W \subset \mathbb{R}^2$ is the union of two opposite angular sectors of common vertex angle $\alpha = \arctan(1/X_0) - \arctan(X_0) \in (0, \pi)$ meeting only at their shared vertex. Intersecting a small sphere $S_\varepsilon^2(p)$ with (interval) $\times W$ gives, for each latitude $x \in (-\varepsilon, \varepsilon)$, two arcs of angular width α on the circle of radius $\sqrt{\varepsilon^2 - x^2}$; sweeping over x , each arc traces out a spherical lune (a bigon from the north pole $x = \varepsilon$ to the south pole $x = -\varepsilon$), and the two lunes share both poles (since the unique point of W at radius 0 is the common vertex). Two disks glued at two distinct boundary points is homotopy equivalent to S^1 (contract each disk to an arc joining its two marked points), giving $\text{Link}(p) \simeq S^1$. Directly: $\chi(2 \text{ disjoint disks}) = 2$, and each of the two point-identifications (gluing the two north poles, then the two south poles) reduces χ by 1, giving $\chi(\text{Link}(p)) = 0$. $\square$

D.4 The theorem

Theorem D.5 (Global Image Theorem). M is homeomorphic to neither the closed 3-ball D^3 nor the cone $C(\mathbb{RP}^2)$. Explicitly, M is four copies of D^3 (the lobes $L_1, \dots, L_4$, Proposition D.1), glued pairwise along six boundary edges (the half-axes, Proposition D.2), with all six edges meeting at the single point O .

Proof. Every point of D^3 has link S^2 (interior) or D^2 (boundary), both with $\chi = 2$ or $\chi = 1$ respectively and $\tilde{H}_1(\cdot; \mathbb{Z}/2) = 0$. By Lemma D.3, $\text{Link}(O) \simeq K_4$ has $\chi = -2$ and $\tilde{H}_1(\text{Link}(O); \mathbb{Z}/2) = (\mathbb{Z}/2)^3 \neq 0$, so O is not a manifold point of M of either type; hence $M \not\cong D^3$.

Every point of $C(\mathbb{RP}^2)$ other than its single apex has link S^2 (a manifold point); only the apex has the exceptional link $\mathbb{RP}^2$. By Lemma D.4, every interior axis point p (a continuum of points, not a single one) has $\text{Link}(p) \simeq S^1$, matching neither S^2 nor $\mathbb{RP}^2$ (indeed $\chi(S^1) = 0 \neq 2 = \chi(S^2)$ and $\neq 1 = \chi(\mathbb{RP}^2)$), so p is a non-manifold point of a type not admitted by $C(\mathbb{RP}^2)$ at any single point, let alone along a continuum. Since a homeomorphism must carry M 's non-manifold locus bijectively onto $C(\mathbb{RP}^2)$'s (a single point), no such homeomorphism exists; $M \not\cong C(\mathbb{RP}^2)$.

The explicit description follows from Propositions D.1 and D.2, which exhibit M as exactly this union. $\square$

Remark D.6. Theorem D.5 fully answers the main topological question: the homeomorphism type of M is pinned down explicitly (four balls glued pairwise along six edges converging on one point), even though this space is not one of the classically named ones initially conjectured. By the isometry of Corollary 11.4 (Appendix C), the identical statement holds for $\text{Image}(q_0, q_1, q_2, q_3)$, with the singular point O corresponding to the maximally depolarizing channel $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and the four lobes corresponding to neighborhoods of the four deterministic Pauli channels.

E Geometric Visualization of the Cosine-Product Image

The cosine-product image is defined by

$$M = \{(\cos(x) \cos(z), \cos(y) \cos(z), \cos(x) \cos(y)) : x, y, z \in [0, 2\pi)\}. \quad (1)$$

Because the cosine function is periodic, the restriction $x, y, z \in [0, 2\pi)$ contains several complete periods and produces the full cosine-product image. The resulting subset of $\mathbb{R}^3$ exhibits four distinguished lobes corresponding to the four admissible sign sectors

$$(+, +, +), \quad (+, -, -), \quad (-, +, -), \quad (-, -, +). \quad (2)$$

The four extremal points are

$$v_1 = (1, 1, 1), \quad v_2 = (1, -1, -1), \quad v_3 = (-1, 1, -1), \quad v_4 = (-1, -1, 1), \quad (3)$$

which are the vertices of the associated tetrahedral geometry.

Figure 1 displays a three-dimensional projection of the image M , together with the tetrahedral vertices and their connecting edges. The figure is intended as a geometric visualization of the four-lobe decomposition established in Section ??.

The visualization is consistent with the semialgebraic description

$$M = \{(X, Y, Z) \in [-1, 1]^3 : XYZ \geq 0, |XY| \leq |Z|, |XZ| \leq |Y|, |YZ| \leq |X|\}. \quad (4)$$

In particular, the four lobes should be understood as the four topological sectors of the global image rather than as four independent smooth manifolds. Their boundaries contain lower dimensional strata, and the four lobes meet through the singular axis structure described in the global topology theorem.

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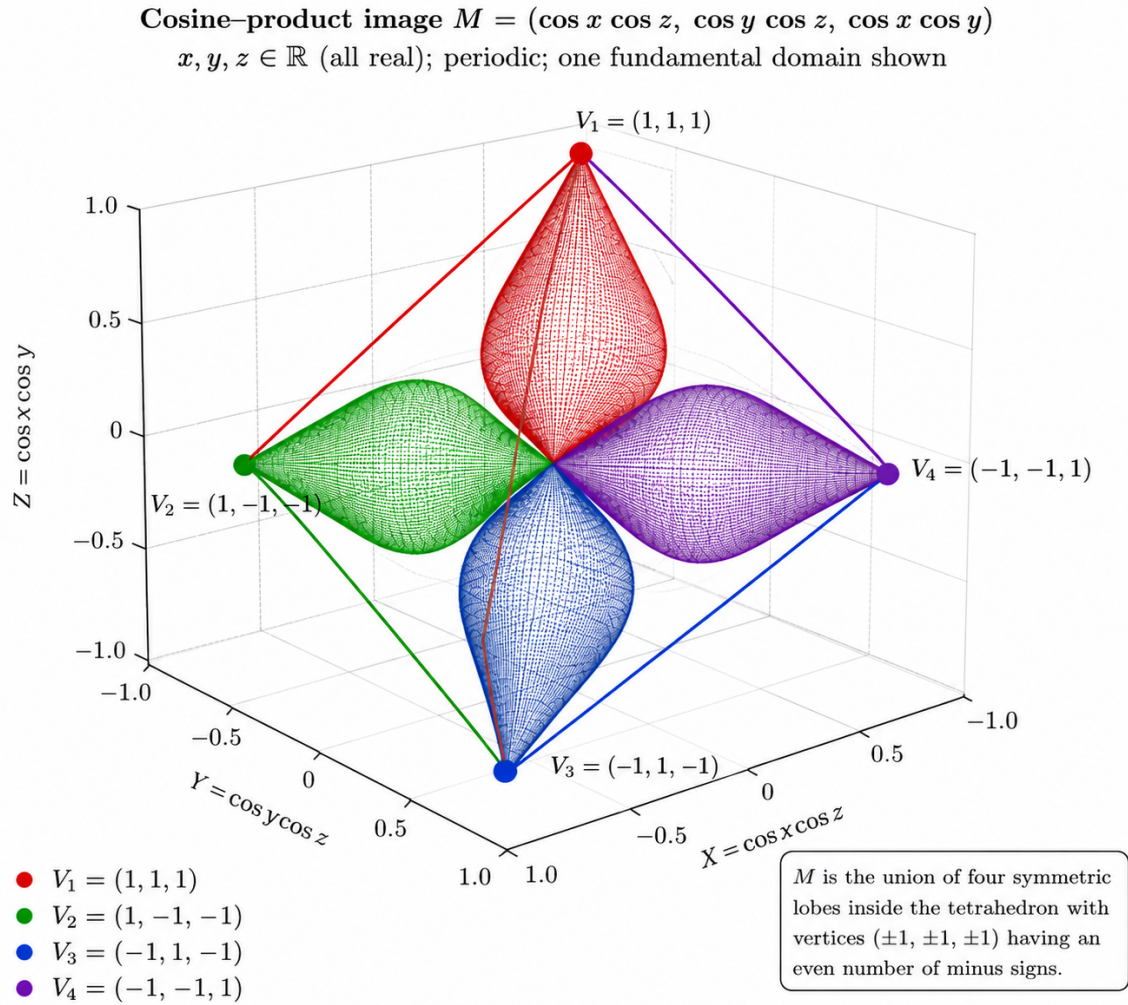


Figure 1: Three-dimensional visualization of the cosine-product image M . The image is organized into four symmetric lobes, directed toward the tetrahedral vertices $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, and $(-1, -1, 1)$.

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