Parameterized Quantum Circuits by Triangle Quantum Channel

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In this paper, we discuss triangle quantum channels, their implementation as parameterized quantum circuits (PQCs), and their applications in variational quantum algorithms (VQAs). According to channel-state duality, there is an one-to-one correspondence between triangle quantum channels and Bell diagonal states (BDSs). We find this connection between a Bloch representation of a triangle quantum channel and a probability generator of Bell diagonal state via a parameterized quantum circuit. Therefore, such parameterized quantum circuits (PQCs) can be used to build general-purpose ansatzs for a variety of applications, such as variational quantum algorithms and quantum machine learning.

I. INTRODUCTION

Channel-state duality is the one-to-one correspondence between quantum channels and bipartite quantum states, which are the states of two quantum systems. The Choi–Jamiołkowski isomorphism Araújo [1] connects these quantum channels (completely positive maps) and quantum states (density matrices). Therefore, quantum channels and unital channels, which contain Pauli channels and triangle channels, correspond to Choi states and Bell diagonal states, respectively.

A quantum state is a mathematical model of the potential results of measuring a quantum system. Quantum channels are related to Choi states, which connect operations and states. To make the Choi state for a channel, we use the maximally entangled state and the channel's effect on it. A Bell diagonal state is a random combination of Bell states, which are the maximally entangled states of two qubits. The partial transpose matrix determines the entanglement of a Bell diagonal state. A Werner state is a mix of a single Bell state and the maximally mixed state I.

A quantum channel is a mathematical model that shows how a quantum system changes under the effect of noise or environment. A unital quantum channel is a quantum channel that keeps the identity matrix unchanged. A Pauli channel is a kind of unital channel that randomly applies a Pauli operator to a qubit with some probability. A triangle channel can send quantum states and information using a set of cosine functions Li [2]. Werner channel is a special unital channel with equal Bloch representation values.

This article shows how we obtained an identity of a probability distribution by applying a Hadamard transform and a triangle qubit channel. We also figure out a parameterized quantum circuit based on the probability distribution, which enabled us to generate a Bell diagonal state using probability generator, vector basis conversion and measurement.

The probability generator of parameterized quantum circuit linked to triangle qubit channel enables us to construct universal ansätze. An ansatz is a German word that means an educated guess or a plausible assumption. Ansätze is the plural form of ansatz. Universal ansätze are a class of parametrized quantum circuits that can be used to approximate any quantum state or unitary operation.

Variational Quantum Algorithms (VQAs) are one of the potential applications of the probability generator with parameterized quantum circuit. They are a class of quantum algorithms that use a hybrid quantum-classical approach to optimize a parameterized quantum circuit or universl ansätze. VQAs can be used to solve various problems, such as finding the ground state energy of a system, solving optimization problems, or performing quantum machine learning.

II. CHANNEL STATE DUALITY

Channel-state duality (CSD) describes the relationship between quantum channels and quantum states. Jiang *et al.* [3] A quantum channel is a mathematical model of a physical process that transforms a quantum state. A quantum state is a mathematical representation of the possible outcomes of measuring a quantum system. Channel-state duality means that there is a one-to-one correspondence between quantum channels and bipartite quantum states, which are quantum states of two quantum systems. This correspondence allows us to use quantum states to study channels, and vice versa.

One way to understand Channel-state duality (CSD) is to use the Choi-Jamiolkowski isomorphism, which is a mathematical map that converts a quantum channel into a quantum state, and vice versa. **?**] The Choi-Jamiolkowski isomorphism works by applying the quantum channel to one half of a maximally entangled state, which is a quantum state that has the highest possible correlation between two quantum systems. The output state is called the Choi matrix, and it encodes the action of the quantum channel on any input state. Conversely, any bipartite quantum state can be converted into a quantum channel by using the inverse of the Choi-Jamiolkowski isomorphism.

A Bell Diagonal State (BDS) is a type of bipartite qubit state that is defined as the probabilistic mixture of Bell states, which are the maximally entangled states of two qubits. **?**] A Bell diagonal state of two qubits can be written in the density

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matrix form as:

$$\rho = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} \lambda_i \sigma_i \otimes \sigma_i) \tag{1}$$

where I is the identity matrix, σ_i are the Pauli matrices, and λ_i are real numbers satisfying $-1 \leq \lambda_i \leq 1$ for i = 1, 2, 3 and $\sum_{i=1}^{3} \lambda_i^2 \leq 1$. The numbers λ_i are called the canonical parameters of the Bell diagonal state, and they are uniquely determined by the state up to permutation.

By Channel-State Duality, there is a one-to-one correspondence between a unital channel and a Bell diagonal state, which means that any unital channel can be represented by a Bell diagonal state, and vice versa.

As a type of unital channel, the Pauli channel is one-to-one correspondence to a Bell diagonal state such that the output state of a Pauli channel applied to a qubit is always a Bell diagonal state, and the canonical parameters of the Bell diagonal state are equal to the probabilities of the Pauli channel. Conversely, any Bell diagonal state can be obtained by applying a Pauli channel to a qubit, and the probabilities of the Pauli channel are equal to the canonical parameters of the Bell diagonal state. Therefore, a Pauli channel and a Bell diagonal state can be seen as two different representations of the same quantum operation or quantum state.

III. TRIANGLE QUANTUM CHANNEL

A. Quantum Channel

A quantum channel is represented by a completely positive and trace-preserving (CPTP) linear map,

$$\Phi : \mathbf{H_A} \to \mathbf{H_B} \tag{2}$$

where $\mathbf{H_A}$ and $\mathbf{H_B}$ are the Hilbert spaces associated with the quantum systems before and after the channel, respectively.

- Linear Map: A quantum channel is a linear map that describes the evolution of a quantum state. If you have a quantum state ρ, the output state after the channel acts on it is given by $\Phi(\rho)$, where Φ is the quantum channel.
- Completely Positive: This property ensures that the quantum channel acts in a way that preserves positivity. If the initial state is a positive operator (density matrix), the final state will also be positive.
- Trace-Preserving: This property ensures that the trace of the density matrix is conserved. The trace represents the probability, and the preservation of trace ensures that probabilities sum to 1.

B. Unital Quantum Channel

A unital quantum channel is a quantum channel that preserves the identity matrix such that $\Phi(I) = I$. A qubit channel Φ is a two-dimensional quantum channel. Any qubit state ρ can be represented by the Pauli matrices σ_i as the basis such that $\rho = \frac{1}{2}\sum_{i=0}^{3} r_i \sigma_i$ where $r_i \in \mathbb{R}$ and $r_0 = 1$. The real vector $\vec{r} = (r_x, r_y, r_z)$ with $trace(\rho) = 1$ is called Bloch vector. The qubit channel Φ acting on the state ρ is a 4 by 4 real matrix T_ϕ. Specially, the unital channel has the matrix

$$T_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} \tag{3}$$

where λ_1, λ_2, and λ_3 are real numbers and $T_\phi = diag(1, \lambda_1, \lambda_2, \lambda_3)$ is called the Bloch representation of the unital channel.

The conditions that the diagonal elements must satisfy completely positive and trace-preserving to ensure that the matrix T_Φ represents a unital qubit channel are:

$$\begin{aligned} q_{00} &= \frac{1}{4}(1 + \lambda_1 + \lambda_2 + \lambda_3) \geq 0 \\ q_{01} &= \frac{1}{4}(1 - \lambda_1 + \lambda_2 - \lambda_3) \geq 0 \\ q_{10} &= \frac{1}{4}(1 + \lambda_1 - \lambda_2 - \lambda_3) \geq 0 \\ q_{11} &= \frac{1}{4}(1 - \lambda_1 - \lambda_2 + \lambda_3) \geq 0 \end{aligned} \tag{4}$$

The series $\{q_{ij}\} \geq 0$ represents the eigenvalues of the Choi matrix of the unital channel. Choi [4] These eigenvalues determine the channel's action on the Bloch sphere, specifically how it scales along the x, y, and z axes.

C. Pauli Channel

A Pauli channel is a type of unital channel that applies a random Pauli operator to a qubit with some probability. A Pauli channel can be written as

$$\Phi(\rho) = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i \tag{5}$$

where $0 \leq p_i \leq 1$ and $\sum_{i=0}^{3} p_i = 1$.

A Pauli operator is a matrix that can flip or rotate a qubit in the Bloch sphere. There are four Pauli operators: $\sigma_0 = I$, σ_1, σ_2, and σ_3, which correspond to the identity, bit-flip, phase-flip, and bit-and-phase-flip operations, respectively. A Pauli channel can be described by a vector of four probabilities (p_0, p_1, p_2, p_3), where p_i is the probability of applying the i-th Pauli operator to the qubit.

The Choi matrix of a Pauli channel is a 4×4 matrix that represents the action of the Pauli channel on one half of a maximally entangled state of two qubits. The Choi matrix of a Pauli channel can be written as:

$$\Phi_C = \frac{1}{4}(I_2 \otimes I_2 + \sum_{i=1}^{3} \lambda_i \sigma_i \otimes \sigma_i) \tag{6}$$

The Kraus matrices of a Pauli channel are a set of matrices that represent the quantum channel in a different way, they are

$$K_0 = p_0 I, K_1 = p_1 \sigma_1, K_2 = p_2 \sigma_2, K_3 = p_3 \sigma_3 \tag{7}$$

And Kraus matrices satisfy the completeness relation $\sum_{i=0}^{3} K_i^\dagger K_i = I$.

D. Triangle Qubit Channel

In the paper Li [2], we have proved a theorem about a unital qubit channels as follows:

Theorem 1. *Given a triangle on the unit circle, there exists a unital qubit channel described by real Bloch representation* $diag(1, \lambda_1, \lambda_2, \lambda_3)$ *where real number* $\lambda_{1,2,3} \in [-1, +1]$ *such that*

$$\begin{gathered}(\lambda_1, \lambda_2, \lambda_3) = \\ (cos(\gamma)cos(\alpha), cos(\gamma)cos(\beta), cos(\beta)cos(\alpha))\end{gathered} \tag{8}$$

where parameters $\alpha, \beta, \gamma \in \mathbb{R}$. *This unital qubit channel is called triangle qubit channel.*

Given vector $\vec{q} = (q_{00}, q_{01}, q_{10}, q_{11})^T$,

$$\vec{q} = \frac{1}{2} H_2 * (1, \lambda_1, \lambda_2, \lambda_3)^T \tag{9}$$

where H_2 is 4x4 Hardamard matrix, the vector $\vec{q}$ is a probability distribution because for each element $q_{ij} \geq 0$ since $\sum_{i,j=0}^{1} q_{ij} = 1$. Assign elements of the vector $\vec{q}$ to Pauli operators as follows,

$$\Phi(\rho) = q_{00}\rho + q_{01}\sigma_1\rho\sigma_1 + q_{11}\sigma_2\rho\sigma_2 + q_{10}\sigma_3\rho\sigma_3 \tag{10}$$

we construct a Pauil channel through the triangle qubit channel.

IV. BELL DIAGONAL STATE AND ENTANGLEMENT

Bell basis is a collection of Bell states. Following the notation in textbook Nielsen and Chuang [5], Bell basis is represented as a matrix of $|\beta_{jk}\rangle$,

$$\begin{pmatrix} |\beta_{00}\rangle & |\beta_{01}\rangle \\ |\beta_{10}\rangle & |\beta_{11}\rangle \end{pmatrix} = \begin{pmatrix} \frac{|00\rangle+|11\rangle}{\sqrt{2}} & \frac{|01\rangle+|10\rangle}{\sqrt{2}} \\ \frac{|00\rangle-|11\rangle}{\sqrt{2}} & \frac{|01\rangle-|10\rangle}{\sqrt{2}} \end{pmatrix}, \tag{11}$$

where $|\beta_{jk}\rangle = \frac{1}{\sqrt{2}}(|0, j > +(-1)^j |1, k \oplus 1 >)$, $j, k \in \{0, 1\}$, and $\oplus$ is the xor.

A. Bell Diagonal States

A Bell diagonal state is defined as the probabilistic mixture of Bell states, which are the maximally entangled states of two qubits. It is written as

$$S_{BD} = \sum_{j,k=0}^{1} q_{jk} |\beta_{jk}\rangle\langle\beta_{jk}| \tag{12}$$

where $\{q_{jk}\}$ is a probablity distribution with $0 \leq q_{jk} \leq 1$ and $\sum_{j,k=0}^{1} q_{jk} = 1$. Given Bloch representation of a unital channel $(\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_{1,2,3} \in [-1, 1]$, since a probability distribution of a unital qubit channel $\Phi_C = diag(1, \lambda_1, \lambda_2, \lambda_3)$ is defined as

$$\begin{pmatrix} q_{00} \\ q_{01} \\ q_{10} \\ q_{11} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \tag{13}$$

where $\sum_{j,k=0}^{1} q_{jk} = 1$ and $0 \leq q_{jk} \leq 1$ for all $j, k \in \{0, 1\}$, Since

$$\begin{aligned}
|\Phi^+\rangle\langle\Phi^+| = |\beta_{00}\rangle\langle\beta_{00}| &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \\
|\Psi^+\rangle\langle\Psi^+| = |\beta_{01}\rangle\langle\beta_{01}| &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
|\Phi^-\rangle\langle\Phi^-| = |\beta_{10}\rangle\langle\beta_{10}| &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \bar{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{1} & 0 & 0 & 1 \end{pmatrix}, \\
|\Psi^-\rangle\langle\Psi^-| = |\beta_{11}\rangle\langle\beta_{11}| &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \bar{1} & 0 \\ 0 & \bar{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{14}$$

where $\bar{1}=-1$, thus we obtain the Bell Diagnoal State (BDS)

$$
\begin{aligned}
S_{BD}(\lambda_1,\lambda_2,\lambda_3) &= \sum_{0\le j,k\le 1} q_{jk}|\beta_{jk}\rangle\langle\beta_{jk}| \\
&= \frac{1}{2}\begin{pmatrix} q_{00}+q_{10} & 0 & 0 & q_{00}-q_{10} \\ 0 & q_{01}+q_{11} & q_{01}-q_{11} & 0 \\ 0 & q_{01}-q_{11} & q_{01}+q_{11} & 0 \\ q_{00}-q_{10} & 0 & 0 & q_{00}+q_{10} \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 1+\lambda_1 & 0 & 0 & \lambda_2+\lambda_3 \\ 0 & 1-\lambda_1 & \lambda_2-\lambda_3 & 0 \\ 0 & \lambda_2-\lambda_3 & 1-\lambda_1 & 0 \\ \lambda_2+\lambda_3 & 0 & 0 & 1+\lambda_1 \end{pmatrix} \\
&= \frac{1}{4}(I_2\otimes I_2+\lambda_2(\sigma_1\otimes\sigma_1)-\lambda_3(\sigma_2\otimes\sigma_2)+\lambda_1(\sigma_3\otimes\sigma_3))
\end{aligned} \tag{15}
$$

B. Entanglement of Bell Diagnoal State

Quantum Entanglement was originally described as a physical phenomenon. In mathematics, it is defined as a quantum state that cannot be represented as a tensor product of basis in Hilbert space.

According to the Positive Partial Transpose (PPT) criterion Peres [6] and Horodecki *et al.* [7], for the positive density matrix $S_{BD}(\lambda_1,\lambda_2,\lambda_3)$ in Eq.(15), if and only if exists a negative eigenvalue of the partial transpose matrix $S_{BD}^{T_B}(\lambda_1,\lambda_2,\lambda_3)$, then the two qubits in the state $S_{BD}(\lambda_1,\lambda_2,\lambda_3)$ is entangled, where

$$
\begin{aligned}
&S_{BD}^{T_B}(\lambda_1,\lambda_2,\lambda_3) \\
&= \frac{1}{4}\begin{pmatrix} 1+\lambda_1 & 0 & 0 & \lambda_2-\lambda_3 \\ 0 & 1-\lambda_1 & \lambda_2+\lambda_3 & 0 \\ 0 & \lambda_2+\lambda_3 & 1-\lambda_1 & 0 \\ \lambda_2-\lambda_3 & 0 & 0 & 1+\lambda_1 \end{pmatrix} \\
&= \frac{1}{4}(I_2\otimes I_2+\lambda_2(\sigma_1\otimes\sigma_1)+\lambda_3(\sigma_2\otimes\sigma_2)+\lambda_1(\sigma_3\otimes\sigma_3))
\end{aligned} \tag{16}
$$

$S_{BD}(\lambda_1,\lambda_2,\lambda_3) = S_{BD}^{T_B}(\lambda_1,\lambda_2,-\lambda_3)$ only one sign difference in λ_3. The eigenvalues of $S_{BD}(\lambda_1,\lambda_2,\lambda_3)$ are $\frac{1}{4}(1+\lambda_1+\lambda_2+\lambda_3), \frac{1}{4}(1-\lambda_1+\lambda_2-\lambda_3), \frac{1}{4}(1+\lambda_1-\lambda_2-\lambda_3), \frac{1}{4}(1-\lambda_1-\lambda_2+\lambda_3)$, while the eigenvalues of $S_{BD}^{T_B}(\lambda_1,\lambda_2,\lambda_3)$ are $\frac{1}{4}(1-\lambda_1-\lambda_2-\lambda_3), \frac{1}{4}(1+\lambda_1-\lambda_2+\lambda_3), \frac{1}{4}(1-\lambda_1+\lambda_2+\lambda_3), \frac{1}{4}(1+\lambda_1+\lambda_2-\lambda_3)$. If all eigenvalues of the state $S_{BD}(\lambda_1,\lambda_2,\lambda_3)$ are non-negative, then, if one of an eigenvalue of $S_{BD}^{T_B}(\lambda_1,\lambda_2,\lambda_3)$ is negative, the state $S_{BD}(\lambda_1,\lambda_2,\lambda_3)$ is entangled.

C. Werner State

Given $\lambda_1=\lambda_2=\lambda_3=\lambda\in[0,1]$, so the eigenvalues of $S_{BD}(\lambda)$ are $q_{00}=(1+3\lambda)/4>0$, $q_{01}=q_{10}=q_{11}=(1-\lambda)/4\ge 0$, where

$$
\begin{aligned}
S_{BD}(\lambda) &= \sum_{j,k=0}^{1} q_{jk}|\beta_{jk}\rangle\langle\beta_{jk}| \\
&= \lambda|\beta_{00}\rangle\langle\beta_{00}| + \frac{1-\lambda}{4}I_4 \\
&= \frac{1}{4}\begin{pmatrix} 1+\lambda & 0 & 0 & 2\lambda \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 2\lambda & 0 & 0 & 1+\lambda \end{pmatrix}
\end{aligned} \tag{17}
$$

$S_{BD}(\lambda)$ is called two-qubit Werner state that is a linear combination of a singlet Bell state and the maximally mixed state I_4.

The eigenvalues of $S_{BD}^{T_B}(\lambda)$ are $(1-3\lambda)/4$ and $(1+\lambda)/4$. Since $(1+\lambda)/4>0$, one negative eigenvalue means $(1-3\lambda)/4<0$. Hence, if $\lambda>1/3$ then the state S_{BD} is entangled.

V. PROBABILITY DISTRIBUTION OF TRINGLE QUBIT CHANNEL

Theorem 2. *Given a triangle qubit channel in Bloch representation*

$$
diag\big(1, cos(\gamma)cos(\alpha), cos(\alpha)cos(\beta), cos(\beta)cos(\gamma)\big) \tag{18}
$$

and

$$
\begin{aligned}
p_{00} &= (cos\frac{\alpha}{2}cos\frac{\beta}{2}cos\frac{\gamma}{2}+sin\frac{\alpha}{2}sin\frac{\beta}{2}sin\frac{\gamma}{2})^2 \\
p_{01} &= (cos\frac{\alpha}{2}cos\frac{\beta}{2}sin\frac{\gamma}{2}-sin\frac{\alpha}{2}sin\frac{\beta}{2}cos\frac{\gamma}{2})^2 \\
p_{10} &= (cos\frac{\alpha}{2}sin\frac{\beta}{2}cos\frac{\gamma}{2}-sin\frac{\alpha}{2}cos\frac{\beta}{2}sin\frac{\gamma}{2})^2 \\
p_{11} &= (cos\frac{\alpha}{2}sin\frac{\beta}{2}sin\frac{\gamma}{2}+sin\frac{\alpha}{2}cos\frac{\beta}{2}cos\frac{\gamma}{2})^2
\end{aligned} \tag{19}
$$

where $\alpha,\beta,\gamma\in\mathbf{R}$ we have the identity below

$$
\begin{aligned}
1 &= p_{00}+p_{01}+p_{10}+p_{11} \\
cos(\gamma)cos(\alpha) &= p_{00}-p_{01}+p_{10}-p_{11} \\
cos(\alpha)cos(\beta) &= p_{00}+p_{01}-p_{10}-p_{11} \\
cos(\beta)cos(\gamma) &= p_{00}-p_{01}-p_{10}+p_{11}
\end{aligned} \tag{20}
$$

Proof. Define

$$
\begin{aligned}
(a,b,c) &= (cos\frac{\alpha}{2}, cos\frac{\beta}{2}, cos\frac{\gamma}{2}) \\
(\bar{a},\bar{b},\bar{c}) &= (sin\frac{\alpha}{2}, sin\frac{\beta}{2}, sin\frac{\gamma}{2})
\end{aligned} \tag{21}
$$

we have

$$a^2+\bar{a}^2=1, a^2-\bar{a}^2=cos(\alpha),$$
$$b^2+\bar{b}^2=1, b^2-\bar{b}^2=cos(\beta),$$
$$c^2+\bar{c}^2=1, c^2-\bar{c}^2=cos(\gamma). \tag{22}$$

and

$$\begin{aligned} p_{00} &= (abc+\bar{a}\bar{b}\bar{c})^2 \\ p_{01} &= (ab\bar{c}-\bar{a}\bar{b}c)^2 \\ p_{10} &= (a\bar{b}c-\bar{a}b\bar{c})^2 \\ p_{11} &= (a\bar{b}\bar{c}+\bar{a}bc)^2 \end{aligned} \tag{23}$$

then the identity is proved as follows,

$$\begin{aligned} 1 &= p_{00}+p_{01}+p_{10}+p_{11} \\ &= (a^2+\bar{a}^2)(b^2+\bar{b}^2) \\ cos(\gamma)cos(\alpha) &= p_{00}-p_{01}+p_{10}-p_{11} \\ &= (c^2-\bar{c}^2)(a^2-\bar{a}^2) \\ cos(\alpha)cos(\beta) &= p_{00}+p_{01}-p_{10}-p_{11} \\ &= (a^2-\bar{a}^2)(b^2-\bar{b}^2) \\ cos(\beta)cos(\gamma) &= p_{00}-p_{01}-p_{10}+p_{11} \\ &= (b^2-\bar{b}^2)(c^2-\bar{c}^2) \end{aligned} \tag{24}$$

Because $0 \le p_{ij} \le 1$ and $\sum_{i,j=0}^{1} p_{ij} = 1$, the set $\{p_{ij}\}$ forms a probability distribution. □

Assign elemetns of the probability distribution $(p_{00}, p_{01}, p_{11}, p_{10})$ above to Pauli operators as follows,

$$\Phi(\rho) = p_{00}\rho + p_{01}\sigma_1\rho\sigma_1 + p_{11}\sigma_2\rho\sigma_2 + p_{10}\sigma_3\rho\sigma_3 \tag{25}$$

we construct a Pauil channel through the triangle qubit channel. The Bell diagonal state resulting from the action of the Pauli channels can be expressed as:

$$\begin{aligned} \rho = p_{00}|\Phi^+\rangle\langle\Phi^+| + p_{01}|\Psi^+\rangle\langle\Psi^+| \\ + p_{10}|\Phi^-\rangle\langle\Phi^-| + p_{11}|\Psi^-\rangle\langle\Psi^-| \end{aligned} \tag{26}$$

VI. QUANTUM CIRCUITS OF TRIANGLE QUBIT CHANNEL

A. Parameterized Probability Generator

In the paper Gårding *et al.* [8] authors presented a quantum circuit of parameterized probability generator with three

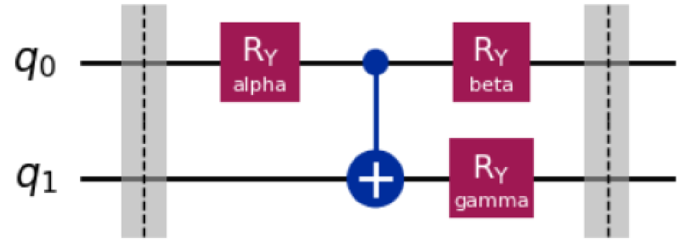

FIG. 1. **Parameterized Probability Generator** The three-phase gates $R_y(\alpha), R_y(\beta), R_y(\gamma)$ and the CNOT gate generate parameterized probability distributions defined in equations (19) and Gårding *et al.* [8].

paramters $\{\alpha, \beta, \gamma\} \in \mathbf{R}$ as follows: where $\theta \in \{\alpha, \beta, \gamma\} \subset \mathbf{R}$, and

$$R_y(\theta) = e^{-i\sigma_2\frac{\theta}{2}} = \begin{pmatrix} cos\frac{\theta}{2} & -sin\frac{\theta}{2} \\ sin\frac{\theta}{2} & cos\frac{\theta}{2} \end{pmatrix} \tag{27}$$

The generated wave function of two qubits with parameterized probability in Eq.(19) is

$$|\Psi_{AB}\rangle = \sum_{j,k=0}^{1} \sqrt{p_{jk}}|jk\rangle \tag{28}$$

B. Generate Bell Diagonal State

By using a Hardmard gate and a CNOT gate along with meausrement, we can make a basis converter from standard basis to Bell basis, then the final Bell Diagonal State (BDS) S_{BD} is

$$\begin{aligned} S_{BD} = \frac{1}{4}(I_2 \otimes I_2 + cos(\alpha)cos(\beta)(\sigma_1 \otimes \sigma_1) \\ + cos(\beta)cos(\gamma)(\sigma_2 \otimes \sigma_2) \\ + cos(\gamma)cos(\alpha)(\sigma_3 \otimes \sigma_3)) \end{aligned} \tag{29}$$

Note there is a substitution of (123) from Bloch representation of triangle qubit channel to quantum circuit respect to Paliu operators above. The negative sign of $-\lambda_3$ has been reflected the product of imaginary number i in $\sigma_2 \otimes \sigma_2$.

VII. APPLICATIONS

A. Variational Quantum Algorithms (VQAs)

Variational Quantum Algorithms (VQAs) are a class of quantum algorithms that use a hybrid quantum-classical approach to optimize a parametrized quantum circuit Tilly

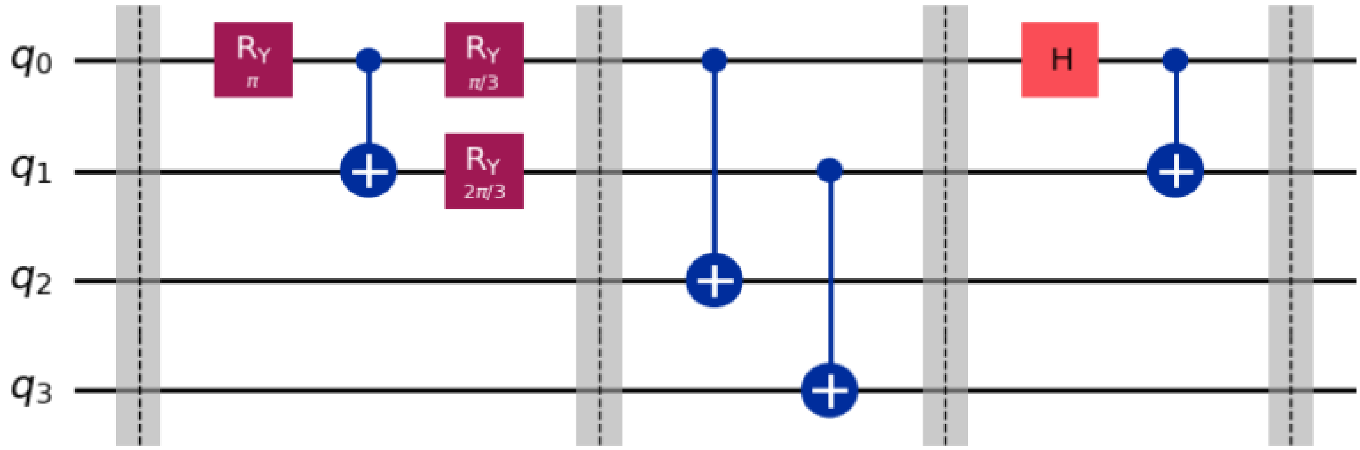

FIG. 2. **Parameterized quantum circuit of Bell Diagonal State.** The first part is a probability generator circuit; the second part replaces the measurement with two CNOT gates; and the third part converts the standard basis to a Bell basis.

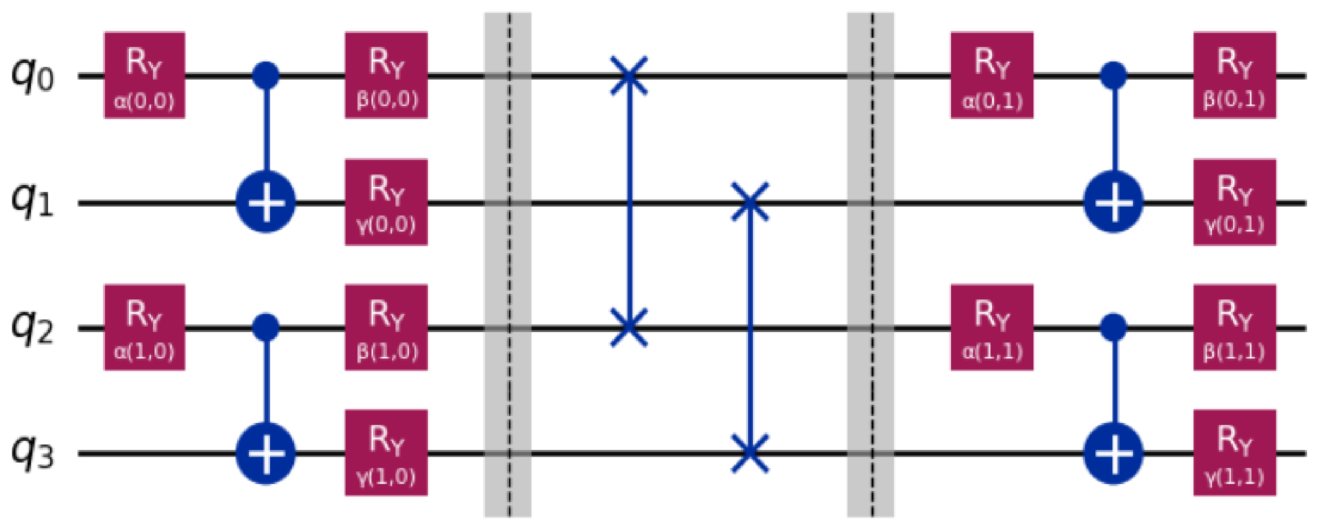

FIG. 3. **PQC of 4 Probability Generators of Triangle Qubit Channel.** The first part is two probability generators. The second part consists of two swap gates to swap dual qubits. The third part also includes two probability generators.

et al. [9]. VQAs represent a promising approach to solving quantum problems by leveraging the variational principle, parameterized quantum circuits, classical optimization routines, and problem-specific objective functions. Optimization or learning based approaches are suitable for NISQ (Noisy Intermediate-scale Quantum) quantum devices that have limited qubits and coherence time. Cerezo *et al.* [10]

1. *Variational Quantum Algorithms (VQAs)*

Variational Quantum Algorithms (VQAs) are based on the variational principle, which states that for any quantum state, the expectation value of the energy is minimized when the state is the exact ground state of the system. Cerezo *et al.* [10]

- Objective Function:
  Depending on the problem, define the objective function that the quantum algorithm aims to minimize. This function is often related to the problem being solved (e.g., energy in quantum chemistry, cost function in machine learning).

- Ansatz and PQCs:
  In VQA, tunable Parameterized Quantum Circuits (PQCs) are used to prepare variational quantum states. These parameters are optimized to minimize the expected value of the chosen objective function. In particular, quantum circuit architectures are specified, including choices of gates, qubits, and entanglement modes.

  Ansatz are state preparation circuits most often de-

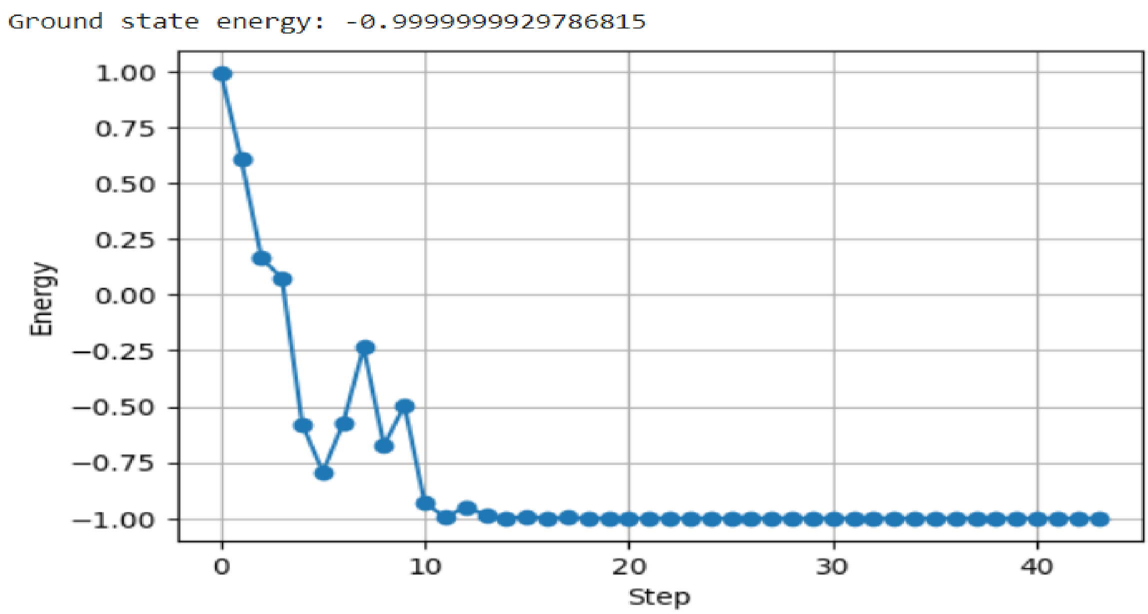

FIG. 4. **A Variational Quantum Eigensolver (VQE) by the PQC of Triangle Qubit Channel** This curve describes a variational quantum eigensolver by using the parameterized quantum circuit of triangle qubit channel with a given Hamiltonian matrix. It shows the process to arrive the ground state energy.

signed to respect the symmetry of the problem Hamiltonian, which helps limit variational searches to the subspace of interest.

Use a quantum computer or quantum simulator to execute a variational quantum circuit and measure the associated observables required to compute the objective function.

- Optimization:
  Classical optimization algorithms are executed to adjust the parameters of the quantum circuit to minimize the objective function. Popular classical optimization algorithms include gradient descent and its variants.

- Iteration:
  Repeat the quantum circuit execution and classical optimization steps iteratively until a satisfactory solution is found.

2. *Variational Quantum Eigensolvers (VQEs)*

Variational Quantum Eigensolvers (VQEs) find the lowest eigenvalues and eigenvectors of a given Hamiltonian and can be used in applications such as quantum chemistry and simulations. VQE seeks to minimize an objective function, which is the expectation value $\langle\Psi(\theta)|H|\Psi(\theta)\rangle$ of H over a trial wave function $|\Psi(\theta)\rangle = U(\theta)|\Psi_0\rangle$ for an ansatz $U(\theta)$ and the initial state $|\Psi_0\rangle$. Tilly *et al.* [9] Qiskit is an open source quantum computing software development framework that supports VQA, which we use to draw quantum circuit diagrams. QiskitDoc [11] Cirq is a Python library for designing, simulating, and running quantum circuits, which we use to implement VQE via a probability generator for triangle quantum channels. CirqDoc [12]

VIII. CONCLUSION

Channel-State Duality (CSD) describes the one-to-one correspondence between quantum channels and bipartite quantum states. This duality allows for the use of quantum states to study channels and vice versa.

Bell diagonal states, which are probabilistic mixtures of Bell states, can be used to represent unital channels. The canonical parameters of the Bell diagonal state correspond to the probabilities of the unital channel.

Triangle quantum channels are represented by completely positive and trace-preserving linear maps. These channels act on quantum states and preserve positivity and the trace of the density matrix.

The probability distribution of a triangle qubit channel can be determined based on the values of α, β, and γ. The probabilities q_{00}, q_{01}, q_{10}, and q_{11} satisfy certain identity involving cosines of α, β, and γ, which is the foundation to link a triangle quantum channel and the parameterized quantum circuit.

Probability generators of triangle quantum channels can be used to parameterize quantum circuits for the applications such as variational quantum algorithms. The basic circuits can be used to build large-scale parameterized quantum circuits for quantum machine learning.

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